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Phil. Trans. R. Soc. Lond. A 1970 **268**, 325-349
doi: 10.1098/rsta.1970.0077

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STOKES MULTIPLIERS FOR THE ORR–SOMMERFELD EQUATION

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(*Communicated by S. Chandrasekhar, F.R.S.—Received* 23 *February* 1970)

CONTENTS

The comparison equation method is used to study the outer expansions of the solutions of the Orr–Sommerfeld equation. All but one of these expansions are multiple-valued and must therefore exhibit the Stokes phenomenon. One of the major aims of the present paper is to obtain first approximations to the Stokes multipliers which describe the continuation of these expansions on crossing a Stokes line in the complex plane. By restricting the domains of validity of these expansions appropriately we can insure that all of the expansions are 'complete' in the sense of Olver and this is an essential feature of the work. The resulting approximations show that, in some sectors, a sharp distinction can no longer be made between approximations of inviscid and viscous type. A consistent first-order approximation to the characteristic equation in the complete sense is derived and compared with the more usual second-order approximation of Poincaré type. Calculations of the curve of neutral stability for plane Poiseuille flow clearly show that a first approximation in the complete sense provides a substantially better approximation to the neutral curve than a second approximation in the Poincaré sense.

1. Introduction

The stability of parallel shear flows is governed by the well-known Orr–Sommerfeld equation

$$(\mathrm{i}\alpha R)^{-1}(\mathrm{D}^2-\alpha^2)^2\phi = (U-c)(\mathrm{D}^2-\alpha^2)\phi - U''\phi, \tag{1.1}$$

where $\phi(y)\,\mathrm{e}^{\mathrm{i}\alpha(x-ct)}$ is the stream function of the disturbance in the usual normal mode analysis, $U(y)$ is the basic velocity distribution, R is the Reynolds number, and $\mathrm{D} = \mathrm{d}/\mathrm{d}y$. In the study of this equation for large values of αR, asymptotic methods of approximation have played an important role and, in the early work on the subject by Heisenberg (1924), Tollmien (1929, 1947), and Lin (1945, 1955), two different types of asymptotic approximations were obtained by somewhat heuristic methods. These approximations correspond to the leading terms of what would now be called inner and outer expansions. Most of the existing calculations, however, have been

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based on the inconsistent but apparently successful procedure of using outer expansions for the solutions of inviscid type and inner expansions for the solutions of viscous type but it is only recently that a detailed study has been made by Eagles (1969) of the limitations of such mixed approximations.

Attempts to improve on these older theories have generally been based on either the comparison equation method or the method of matched asymptotic expansions, and it is of some importance to discuss briefly the essential differences between these two approaches. In applying the method of matched asymptotic expansions to the Orr–Sommerfeld equation, as Eagles (1969) has recently done in a very systematic manner, one is primarily concerned with what may be called the *central matching problem* (cf. Wasow 1968), i.e. the problem of relating the inner and outer expansions so that they represent different asymptotic approximations to the same solutions. Since all but one of the outer expansions are multiple-valued, however, the solutions which they represent must exhibit the Stokes phenomenon and this leads to a consideration of the *lateral connexion problem*, i.e. the problem of determining the continuation of a given solution (or, more precisely, its outer expansion) on crossing a Stokes line in the complex plane. These two problems are closely related and, although Eagles (1969) did not consider the lateral connexion problem explicitly, it should be emphasized that a complete solution to the central matching problem must necessarily contain the solution to the lateral connexion problem but not conversely.

By using the comparison equation method, however, it is possible to determine the Stokes multipliers in an indirect manner that avoids the need for a complete solution of the central matching problem, and this is the approach that will be adopted in the present paper. The comparison equation method has been extensively studied by Wasow (1953), Langer (1957, 1958), Lin (1957*a*, *b*, 1958), Lin & Rabenstein (1960) and others. In all of this work the major aims have been to obtain asymptotic approximations to the solutions of the Orr–Sommerfeld equation that are uniformly valid in a bounded domain containing one critical point and to develop an algorithm by which higher approximations can be systematically obtained. Theories of this type are largely based on the idea of generalizing Langer's (1932) well-known theory for second-order differential equations with a simple turning point to higher-order equations of the Orr–Sommerfeld type. This requires the development of a procedure by which the solutions of the Orr–Sommerfeld equation can be represented asymptotically in terms of the solutions of a suitably chosen comparison equation. The success of this method, however, crucially depends upon being able to satisfy two closely related conditions. First, the comparison equation must be sufficiently simple so that its solutions may be considered known, otherwise little would be achieved; and, secondly, the solutions of the comparison equation must have asymptotic properties that are close to those of the Orr–Sommerfeld equation in order to achieve the desired degree of uniformity in the resulting approximations. These conditions severely limit the class of flows for which the general theory has thus far been developed and would appear to exclude, for example, asymmetrical flows with two critical points.

Although the present paper is closely related to the work of Lin (1957*a*, *b*, 1958) and Lin & Rabenstein (1960), our work differs from theirs in some important respects. The most significant difference results from our insistence that all asymptotic expansions be 'complete' in the sense of Olver (1961, 1963, 1964). The concept of a complete asymptotic expansion, as developed by Olver in connexion with his theory of error bounds for asymptotic solutions of certain second-order differential equations, is based on the observation that it is often more important to obtain a first approximation that is valid in the complete sense than to obtain the whole of the descending

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series associated with the dominant term in the expansion. One consequence of this concept is that different asymptotic expansions of a given solution must be restricted to non-overlapping domains, the boundaries of which are Stokes lines, even though the expansions remain valid in the Poincaré sense in larger overlapping domains.

Thus, the major aims of the present paper are to obtain first approximations to the Stokes multipliers and then, using outer expansions only, to derive a consistent first approximation to the characteristic equation that is valid in the complete sense.

2. Transformation to standard form

A preliminary transformation of the Orr–Sommerfeld equation will first be made which brings out explicitly the turning point nature of the problem. For this purpose we define the usual Langer variable

$$\eta = \left[\frac{3}{2}\int_{y_c}^{y}\left(\frac{U-c}{U_c'}\right)^{\frac{1}{2}} \mathrm{d}y\right]^{\frac{2}{3}}, \tag{2.1}$$

where y_c is a simple zero of $U-c$ and $U_c' \equiv U'(y_c)$. Near $y = y_c$, we have

$$\eta = y-y_c+\frac{1}{10}\frac{U_c''}{U_c'}(y-y_c)^2+\left(\frac{1}{42}\frac{U_c'''}{U_c'}-\frac{2}{175}\frac{U_c''^2}{U_c'^2}\right)(y-y_c)^3+\dots, \tag{2.2}$$

so that $\eta(y)$ is analytic at $y = y_c$. Thus, if D_y denotes a bounded domain in the y-plane containing the critical point y_c, then the relation (2.1) maps D_y on a bounded domain D_η in the η-plane containing the origin. We next define a new dependent variable $\chi(\eta)$ by the relation

$$\chi(\eta) = \{\eta'(y)\}^{+\frac{3}{2}}\,\phi(y), \tag{2.3}$$

where the exponent of $\eta'(y)$ has been chosen so that the transformed equation is in normal form, i.e. it does not contain χ'''. This definition of $\chi(\eta)$ is the one that is most commonly made but it is clearly not essential and for some purposes a different choice might be preferable.

Under this change of both dependent and independent variables, the Orr–Sommerfeld equation becomes

$$\epsilon^3\chi^{\mathrm{iv}}-(\eta+\epsilon^3 f_1)\,\chi''-(g_0+\epsilon^3 g_1)\,\chi'-(h_0+\epsilon^3 h_1)\,\chi = 0, \tag{2.4}$$

where

$$\epsilon = (\mathrm{i}\alpha R U_c')^{-\frac{1}{3}} \tag{2.5}$$

and $f_1, g_0, \dots, h_1$ are all analytic functions of η in D_η. If, for convenience, we let

$$\gamma(\eta) = \eta''/\eta'^2, \tag{2.6}$$

then we have

$$\left.\begin{aligned} f_1(\eta) &= \tfrac{5}{2}\gamma^2+5\gamma'+2\alpha^2\eta'^{-2},\\ g_0(\eta) &= -2\eta\gamma,\\ g_1(\eta) &= 5(\gamma\gamma'+\gamma'')-4\alpha^2\gamma\eta'^{-2},\\ h_0(\eta) &= -(5\gamma+\tfrac{21}{4}\eta\gamma^2+\tfrac{7}{2}\eta\gamma'+\alpha^2\eta\eta'^{-2})\\ \text{and}\quad h_1(\eta) &= -(\tfrac{9}{16}\gamma^4+\tfrac{9}{4}\gamma^2\gamma'+\tfrac{3}{4}\gamma'^2-\tfrac{3}{2}\gamma\gamma''-\tfrac{3}{2}\gamma''')+\alpha^2(\tfrac{3}{2}\gamma^2-3\gamma')\eta'^{-2}-\alpha^4\eta'^{-4}.\end{aligned}\right\} \tag{2.7}$$

The behaviour of these functions near $\eta = 0$ is of crucial importance in the subsequent analysis and we note, therefore, that

$$\left.\begin{aligned} g_0(0) &= 0, & g_0'(0) &= -2\gamma_0, & h_0(0) &= -5\gamma_0,\\ f_1(0) &\neq 0, & g_1(0) &\neq 0, & h_1(0) &\neq 0,\end{aligned}\right\} \tag{2.8}$$

where

$$\gamma_0 \equiv \gamma(0) = \tfrac{1}{5}(U_c''/U_c'). \tag{2.9}$$

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Equation (2.4) is the required standard form of the transformed Orr–Sommerfeld equation and provides the natural starting-point for the present analysis. We assume, as usual, that when y is real $U(y)$ is monotone increasing. In the case of neutral stability for which c is real, we then choose $\operatorname{ph} U_c' = 0$ and $\operatorname{ph} \epsilon = -\frac{1}{6}\pi$.

3. Formal construction of asymptotic solutions

The principle on which we now seek to construct asymptotic approximations to the solutions of equation (2.4) is to express the solutions of this equation asymptotically in terms of the solutions of a suitably chosen comparison equation, the solutions of which are considered known. In this approach there are then two closely related questions that must be considered. One concerns the choice of the comparison equation and the other the general form of the expansion.

Following Lin & Rabenstein (1960) we choose a comparison equation of the form†

$$\epsilon^3 u^{\mathrm{iv}} - (\eta u'' + \alpha u' + \beta u) = 0 \tag{3.1}$$

in which α and β must be allowed to be asymptotic power series in ϵ of the form

$$\alpha = \sum_{n=0}^{\infty} \alpha_n \epsilon^{3n} \quad \text{and} \quad \beta = \sum_{n=0}^{\infty} \beta_n \epsilon^{3n}. \tag{3.2}$$

A cursory comparison of equation (3.1) with equation (2.4) suggests that

$$\alpha_0 = g_0(0) = 0 \quad \text{and} \quad \beta_0 = h_0(0) = -U_c''/U_c'. \tag{3.3}$$

These are, in fact, the correct choices for α_0 and β_0; more generally, however, the coefficients in the expansions (3.2) are determined by certain regularity conditions which will be discussed shortly. The solutions of the comparison equation (3.1) have been discussed previously by Rabenstein (1958) and they are discussed further in the appendix in a form more suitable for the present purposes.

Consider next the problem of representing the solutions of equation (2.4) in terms of the solutions of equation (3.1). This problem has also been discussed by Lin & Rabenstein (1960) who showed that the required form of the expansion can be deduced from the fact that the reduced forms of equations (2.4) and (3.1) obtained by formally letting $\epsilon \to 0$ are both of second order. More generally, as Langer (1957) has observed, equations (2.4) and (3.1) have at least one formal solution which is a power series in ϵ^3 with coefficients that are analytic in D_η. This observation then suggests that the required expansion must be of the form

$$\chi = Au + Bu' + \epsilon^3 (Cu'' + Du'''), \tag{3.4}$$

where u is a solution of the comparison equation and the coefficients A, B, C and D are all asymptotic power series in ϵ of the form

$$A = \sum_{n=0}^{\infty} A_n(\eta)\, \epsilon^{3n}. \tag{3.5}$$

The success of this method depends, of course, on our being able to choose the constants α_n and β_n $(n = 0, 1, 2, \ldots)$ so that all of the coefficients in the expansion (3.4) are analytic in D_η. The form of the expansion actually used by Lin & Rabenstein (1960) differs somewhat from (3.4), the major difference being that they expanded u in terms of χ and its first three derivatives. The

† The α appearing in this equation is not related to the wavenumber.

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more direct form of the expansion adopted here would appear to be more natural and simplifies the comparison of the present results with those obtained by other methods of approximation.

To determine the coefficients in the expansion, we first substitute equation (3.4) directly into equation (2.4) and use the comparison equation to eliminate u^{iv}. The requirement that the coefficients of u, u', u'', and u''' all vanish then leads to the equations

$$u: \quad R_2 A = \beta(4B' + 2\eta D' + 3D + \alpha D - g_0 D) + (\beta - \beta_0) A + \epsilon^3\{A^{\text{iv}} + 6\beta C'' + 4\beta D''' - f_1(A'' + \beta C + 2\beta D') - g_1(A' + \beta D) - h_1 A\}, \tag{3.6}$$

$$u': \quad R_2 B - 4\alpha B' = -2\eta A' - g_0 A + \alpha(A + 2\eta D' + 3D + \alpha D - g_0 D) + (\beta - \beta_0) B + \epsilon^3\{4A''' + B^{\text{iv}} + 6\alpha C'' + 4\beta C' + 4\alpha D'' + 6\beta D'' - f_1(2A' + B'' + \alpha C + 2\alpha D' + \beta D) - g_1(A + B' + \alpha D) - h_1 B\}, \tag{3.7}$$

$$u'': \quad 2\eta B' + (1 - g_0) B + 2\eta^2 D' + (3 - g_0)\eta D = -\alpha(B + \eta D) - \epsilon^3\{6A'' + 4B''' + 5\eta C'' + 4C' + 4\alpha C' - g_0 C' + \beta C + 4\eta D''' + 6D'' + 6\alpha D'' + 4\beta D' - h_0 C - f_1(A + 2B' + \eta C + 2\eta D' + D + \alpha D) - g_1(B + \eta D)\} - \epsilon^6 (C^{\text{iv}} - f_1 C'' - g_1 C' - h_1 C), \tag{3.8}$$

$$u''': \quad 4A' + 6B'' + 2\eta C' + 2C + 5\eta D'' + 8D' + \beta D = -\alpha(C + 4D') + g_0(C + D') + h_0 D + f_1(B + \eta D) - \epsilon^3\{4C''' + D^{\text{iv}} - f_1(2C' + D'') - g_1(C + D') - h_1 D\}, \tag{3.9}$$

where
$$R_2 = \eta \frac{\mathrm{d}^2}{\mathrm{d}\eta^2} + g_0(\eta) \frac{\mathrm{d}}{\mathrm{d}\eta} + \{h_0(\eta) - \beta_0\}. \tag{3.10}$$

We next substitute into these equations the expansions for A, B, C and D and, to lowest order, we obtain

$$R_2 A_0 = \beta_0(4B_0' + 2\eta D_0' + 3D_0 + \alpha_0 D_0 - g_0 D_0), \tag{3.11}$$

$$R_2 B_0 - 4\alpha_0 B_0' = -2\eta A_0' - g_0 A_0 + \alpha_0(A_0 + 2\eta D_0' + 3D_0 + \alpha_0 D_0 - g_0 D_0), \tag{3.12}$$

$$2\eta B_0' + (1 - g_0) B_0 + 2\eta^2 D_0' + (3 - g_0) \eta D_0 = -\alpha_0 (B_0 + \eta D_0) \tag{3.13}$$

and
$$4A_0' + 6B_0'' + 2\eta C_0' + 2C_0 + 5\eta D_0'' + 8D_0' + \beta_0 D_0 = -\alpha_0(C_0 + 4D_0') + g_0 (C_0 + D_0') + h_0 D_0 + f_1(B_0 + \eta D_0). \tag{3.14}$$

Even these first-order equations have a somewhat complicated structure but, fortunately, they can be substantially simplified and partially uncoupled.

Thus, consider equation (3.13) which can be integrated immediately to give

$$B_0 + \eta D_0 = \text{constant}\, \eta^{-\frac{1}{2}(1+\alpha_0)} \eta'^{-1}. \tag{3.15}$$

Since B_0 and D_0 must both be analytic at $\eta = 0$, the constant in this equation must vanish unless $\alpha_0 = -3, -5, \ldots$. Even these special values of α_0 can be excluded; for, if we require as we must that the expansion of χ' has the same general form as equation (3.4) then we find that $B + \eta D$ must be of order ϵ^3 and hence, quite generally, that

$$B_0 + \eta D_0 = 0. \tag{3.16}$$

If this relation is now used to eliminate B_0 from equations (3.11) and (3.12) then we obtain two coupled equations for A_0 and D_0 the solutions of which can be analytic at $\eta = 0$ if and only if we choose

$$\alpha_0 = 0 \quad \text{and} \quad \beta_0 = h_0(0) = -U_c''/U_c' \tag{3.17}$$

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in agreement with equations (3.3). The equations for A_0 and D_0 thus simplify to

$$R_2 A_0 = -\beta_0\{2\eta D_0' + (1+g_0) D_0\} \tag{3.18}$$

and

$$R_2(\eta D_0) = 2\eta A_0' + g_0 A_0. \tag{3.19}$$

The role of the operator R_2 in these equations is of particular interest; for, with the choice of β_0 that we have made, R_2 comes from transformation of the operator

$$\frac{U-c}{U_c'}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \alpha^2\right) - \frac{U'' - U_c''}{U_c'}, \tag{3.20}$$

which is simply a regularized form of the Rayleigh stability operator.

In discussing the solutions of the equations (3.18) and (3.19) we shall require, of course, that A_0 and D_0 both be analytic at $\eta = 0$. It then follows immediately that A_0 is of order unity and D_0 is of order η near $\eta = 0$. Without loss of generality we can choose $A_0(0) = 1$ and these equations then admit a one parameter family of solutions in which the parameter may conveniently be taken as $A_0'(0)$. Again, without loss of generality but with considerable simplification, we can choose $A_0'(0) = 0$. The values of $A_0(0)$ and $A_0'(0)$ chosen here are related to, and consistent with, the normalization conditions that will be imposed later on the 'inviscid' solutions of equations (2.4) and (3.1). With these conditions we then have

$$A_0(\eta) = 1 + \left(\frac{17}{28}\frac{U_c'''}{U_c'} - \frac{1157}{1400}\frac{U_c''^2}{U_c'^2} + \tfrac{1}{2}\alpha^2\right)\eta^2 + O(\eta^3) \tag{3.21}$$

and

$$D_0(\eta) = -\frac{1}{5}\frac{U_c''}{U_c'}\eta + \left(\frac{5}{14}\frac{U_c'''}{U_c'} - \frac{1109}{2100}\frac{U_c''^2}{U_c'^2} + \tfrac{1}{3}\alpha^2\right)\eta^2 + O(\eta^3). \tag{3.22}$$

These results could, if required, be used to provide starting values for the direct numerical integration of equations (3.18) and (3.19). As Lin (1958) has shown, however, A_0 and D_0 can both be expressed explicitly in terms of the 'inviscid' solutions of equations (2.4) and (3.1), though the existence of such a representation is by no means obvious merely from an inspection of the differential equations (3.18) and (3.19).

On using these results in equation (3.14) and then eliminating D_0'' by using equation (3.19) we obtain

$$\left(\frac{\mathrm{d}}{\mathrm{d}\eta} + \gamma\right)(\eta C_0 + A_0 - D_0) = 0. \tag{3.23}$$

This equation can, of course, be immediately integrated and the condition that C_0 be analytic at $\eta = 0$ then gives

$$C_0 = \frac{1}{\eta}\left(\frac{1}{\eta'} - A_0 + D_0\right). \tag{3.24}$$

We have thus succeeded in reducing the set of first-order equations (3.11) to (3.14) to the pair of coupled differential equations (3.18) and (3.19) for A_0 and D_0, with B_0 and C_0 given explicitly in terms of A_0 and D_0 by equations (3.16) and (3.24).

There is, of course, no difficulty in writing down the higher-order sets of equations for A_n, B_n, C_n and D_n ($n \geqslant 1$) that follow from equations (3.6) to (3.9) but they rapidly become far too complicated for complete analysis. In the present theory, however, B_1 is required (but not A_1, C_1 or D_1) and we shall therefore consider only briefly the set of equations of second order. Thus, from equations (3.6) to (3.9) we have

$$\begin{aligned} R_2 A_1 = {} & \beta_0(4B_1' + 2\eta D_1' + 3D_1 + \alpha_1 D_0 - g_0 D_1) + \beta_1 A_0 + A_0^{\mathrm{iv}} + 6\beta_0 C_0'' + 4\beta_0 D_0''' \\ & - f_1(A_0'' + \beta_0 C_0 + 2\beta_0 D_0') - g_1(A_0' + \beta_0 D_0) - h_1 A_0, \end{aligned} \tag{3.25}$$

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$$R_2 B_1 = -2\eta A_1' - g_0 A_1 + \alpha_1(A_0 + 4B_0' + 2\eta D_0' + 3D_0 - g_0 D_0) + \beta_0 B_0 + 4A_0''' + B_0^{\mathrm{iv}} + 4\beta_0 C_0' + 6\beta_0 D_0'' - f_1(2A_0' + B_0'' + \beta_0 D_0) - g_1(A_0 + B_0') - h_1 B_0, \tag{3.26}$$

$$2\eta B_1' + (1-g_0) B_1 + 2\eta^2 D_1' + (3-g_0)\eta D_1 = -(6A_0'' + 4B_0''' + 5\eta C_0'' + 4C_0' - g_0 C_0' + \beta_0 C_0 + 4\eta D_0''' + 6D_0'' + 4\beta_0 D_0' - h_0 C_0) + f_1(A_0 + 2B_0' + \eta C_0 + 2\eta D_0' + D_0) \tag{3.27}$$

and

$$4A_1' + 6B_1'' + 2\eta C_1' + 2C_1 + 5\eta D_1'' + 8D_1' + \beta_0 D_1 + \beta_1 D_1 = -\alpha_1(C_0 + 4D_0') + g_0(C_1 + D_1') + h_0 D_1 + f_1(B_1 + \eta D_1) - (4C_0''' + D_0^{\mathrm{iv}}) + f_1(2C_0' + D_0'') + g_1(C_0 + D_0') + h_1 D_0. \tag{3.28}$$

In spite of the complicated appearance of these equations, their general structure is very similar to that of equations (3.11) to (3.14). Thus, equation (3.27) can be immediately integrated and the condition that B_1 and D_1 be analytic at $\eta = 0$ then gives

$$B_1 + \eta D_1 = \frac{1}{2\eta^{\frac{1}{2}}\eta'} \int_0^{\eta} \frac{\eta'}{\eta^{\frac{1}{2}}} M_0(\eta)\, d\eta, \tag{3.29}$$

where $M_0(\eta)$ denotes the right-hand side of equation (3.27). The quantity $M_0(\eta)$ is analytic in D_η and depends only on the known coefficients A_0, B_0, C_0, D_0 and their derivatives.

If equation (3.29) were now used to eliminate B_1 from equations (3.25) and (3.26) then we would obtain two coupled equations for A_1 and D_1, the solutions of which can be analytic at $\eta = 0$ if and only if α_1 and β_1 are properly chosen. The required value of α_1 can be found without difficulty and it is perhaps of some interest to do so even though it is not required in the present theory. For this purpose consider equation (3.26). Since A_1 and B_1 must both be analytic at $\eta = 0$, the constant term in the power series expansion of the right-hand side of that equation must vanish and this is the condition that determines α_1. Thus, we have

$$\alpha_1 A_0(0) + 4A_0'''(0) + B_0^{\mathrm{iv}}(0) + 4\beta_0 C_0'(0) + 6\beta_0 D_0''(0) - f_1(0)\{2A_0'(0) + B_0''(0)\} - g_1(0) A_0(0) = 0, \tag{3.30}$$

which shows immediately that the value of α_1 is independent of $h_1(0)$, i.e. it does not depend on U_c^{v}. At first glance it might appear from this equation that α_1 depends on the way in which we have normalized $A_0(\eta)$. A somewhat more detailed study shows, however, that the value of α_1 is indeed invariant with respect to the normalization of $A_0(\eta)$. A straightforward but lengthy calculation then gives

$$\alpha_1 = -\frac{U_c^{\mathrm{iv}}}{U_c'} + \frac{U_c''' U_c''}{U_c'^2}. \tag{3.31}$$

This unexpectedly simple and beautiful result shows that α_1 is independent of α^2 and that it vanishes for both plane Poiseuille flow and the asymptotic suction boundary-layer profile. The corresponding determination of β_1, however, would appear to be much more difficult (in part because β_1 depends on U_c^{vi}), and we have found no simple way of determining its value.

From the foregoing discussion it is clear that, in this method of determining the coefficients in the expansion, B_1 cannot be found without first solving the coupled differential equations satisfied by A_1 and D_1. Once this has been done, however, C_1 can then be obtained immediately from equation (3.28) by means of a single quadrature. It is also of some interest to notice that the value of $B_1(0)$ can easily be obtained. Thus, on setting $\eta = 0$ in equation (3.27) we have

$$B_1(0) = -6A_0''(0) + f_1(0) A_0(0) - 4B_0'''(0) - 4C_0'(0) - 6D_0''(0) - 4\beta_0 D_0'(0) \tag{3.32}$$

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and a short calculation then gives

$$B_1(0) = -\frac{U_c''''}{U_c'} + \frac{17}{30}\frac{U_c''^2}{U_c'^2} + \tfrac{2}{3}\alpha^2. \tag{3.33}$$

An explicit determination of B_1 will be obtained later in §6 by a different method which avoids any reference to A_1, C_1 or D_1.

4. Approximations to the inviscid solutions of Tollmien type

The coefficients in the expansion (3.4) can also be determined by matching the uniform approximations in suitably restricted domains of the η-plane to outer expansions of the more familiar type. This indirect method was first suggested by Lin (1957*a*) and leads to a somewhat more explicit representation of the coefficients in the expansion. It also avoids the complicated sets of coupled differential equations that occur naturally in the direct method described in the previous section. The required outer expansions are of the usual inviscid and viscous types but we shall restrict their domains of validity so that they are complete in the sense of Olver. Furthermore, to derive a consistent approximation to the characteristic equation it is necessary to obtain these outer expansions to a somewhat higher order of approximation than is usually done.

Consider then the inviscid approximations which are obtained by a formal expansion of the form

$$\chi(\eta) = \chi^{(0)}(\eta) + \epsilon^3\chi^{(1)}(\eta) + \dots. \tag{4.1}$$

To first order we have

$$(\eta \mathrm{D}^2 + g_0 \mathrm{D} + h_0)\chi^{(0)} = 0, \tag{4.2}$$

where $\mathrm{D} = \mathrm{d}/\mathrm{d}\eta$. This is simply Rayleigh's stability equation written in terms of the new variables, and its solutions can conveniently be written in the form

$$\chi_1^{(0)}(\eta) = \eta Q_1(\eta), \tag{4.3}$$

and

$$\chi_2^{(0)}(\eta) = Q_2(\eta) + (U_c''/U_c')\chi_1^{(0)}(\eta)\ln\eta, \tag{4.4}$$

where $Q_1(\eta)$ and $Q_2(\eta)$ are power series in η with leading terms of unity. To make $\chi_2^{(0)}(\eta)$ definite we suppose, as usual, that $Q_2(\eta)$ contains no multiple of $\chi_1^{(0)}(\eta)$, i.e. that the coefficient of the linear term in $Q_2(\eta)$ is zero. A simple calculation then gives

$$Q_1(\eta) = 1 + \frac{7}{10}\frac{U_c''}{U_c'}\eta + \left(\frac{1}{4}\frac{U_c'''}{U_c'} - \frac{3}{200}\frac{U_c''^2}{U_c'^2} + \tfrac{1}{6}\alpha^2\right)\eta^2 + \dots \tag{4.5}$$

and

$$Q_2(\eta) = 1 + \left(\frac{17}{28}\frac{U_c'''}{U_c'} - \frac{1927}{1400}\frac{U_c''^2}{U_c'^2} + \tfrac{1}{2}\alpha^2\right)\eta^2\dots. \tag{4.6}$$

With this normalization, $\chi_1^{(0)}(\eta)$ and $\chi_2^{(0)}(\eta)$ are related to the corresponding solutions $\phi_1^{(0)}(y)$ and $\phi_2^{(0)}(y)$ of Rayleigh's stability equation by the relations

$$\phi^{(0)}(y) = \{\eta'(y)\}^{-\frac{3}{2}}\chi_1^{(0)}(\eta) \tag{4.7}$$

and

$$\phi_2^{(0)}(y) = \{\eta'(y)\}^{-\frac{3}{2}}\left\{\chi_2^{(0)}(\eta) + \frac{3}{10}\frac{U_c''}{U_c'}\chi_1^{(0)}(\eta)\right\}, \tag{4.8}$$

where $\phi_1^{(0)}(y)$ and $\phi_2^{(0)}(y)$ have their usual normalization (cf. Reid 1965). We also note that the Wronskian of these solutions is a constant with the value

$$W(\chi_1^{(0)}, \chi_2^{(0)}) = -1. \tag{4.9}$$

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It is also of some interest to consider briefly the second approximation to χ_1. Thus, to second order we have

$$(\eta D^2 + g_0 D + h_0)\chi_1^{(1)} = (D^4 - f_1 D^2 - g_1 D - h_1)\chi_1^{(0)}. \tag{4.10}$$

We require that the solution of this equation be analytic at $\eta = 0$ thereby excluding a multiple of $\chi_2^{(0)}$ and, to fix the normalization, we suppose that it contains no multiple of $\chi_1^{(0)}$. A short calculation then gives

$$\chi_1^{(0)}(\eta) = -(U_c^{\mathrm{iv}}/U_c'') + \tfrac{2}{3}\alpha^2 + O(\eta^2). \tag{4.11}$$

Since $\chi_2^{(0)}(\eta)$ has a logarithmic branch point at $\eta = 0$, it cannot provide a uniform approximation in a closed sector of angle 2π and its domain of validity is normally taken to be the sector $-\frac{7}{6}\pi < \mathrm{ph}\,\eta < \frac{1}{6}\pi$. In parts of this sector, however, it is not complete in the sense of Olver and we shall therefore further restrict its domain of validity to the smaller sector $-\frac{5}{6}\pi < \mathrm{ph}\,\eta < -\frac{1}{6}\pi$; it does, of course, remain valid in the ordinary Poincaré sense in the larger sector. Since this smaller sector contains neither of the boundary points, it will therefore be necessary to obtain the continuation of $\chi_2^{(0)}(\eta)$ across the lines which bound its sector of validity. This will be done later in §7 by use of the comparison equation.

5. Approximations to the viscous solutions of W. K. B. type

Consider next the W.K.B. approximations to the viscous solutions of the Orr–Sommerfeld equation which can be derived in the usual way by letting

$$\phi(\eta) = \exp\left\{\int g(y)\,\mathrm{d}y\right\}, \tag{5.1}$$

so that $g(y)$ satisfies the third-order nonlinear equation

$$g^4 + 6g^2g' + 4gg'' + 3g'^2 + g''' - 2\alpha^2(g^2+g') + \alpha^4 = \epsilon^{-3}\left\{\left(\frac{U-c}{U_c'}\right)(g^2+g'-\alpha^2) - \frac{U''}{U_c'}\right\}. \tag{5.2}$$

Approximations to the solutions of this equation are then obtained by assuming a formal expansion of the form

$$g(y) = \epsilon^{-\frac{3}{2}}g_0(y) + g_1(y) + \epsilon^{\frac{3}{2}}g_2(y) + \dots. \tag{5.3}$$

In the usual discussions of approximations of this type only g_0 and g_1 are determined from which first approximations of W.K.B. type are then obtained. In the present theory, however, we need second approximations of this type and we must therefore also determine g_2. If the expansion (5.3) is now formally substituted into equation (5.2) and the coefficients of like powers of $\epsilon^{\frac{3}{2}}$ are equated to zero then we obtain a sequence of equations from which $g_0, g_1, \dots$ can be determined algebraically. The first three equations in this sequence are

$$g_0^2\left(g_0^2 - \frac{U-c}{U_c'}\right) = 0, \tag{5.4}$$

$$2g_0\left(2g_0^2 - \frac{U-c}{U_c'}\right)g_1 = -\left(6g_0^2 - \frac{U-c}{U_c'}\right)g_0', \tag{5.5}$$

$$2g_0\left(2g_0^2 - \frac{U-c}{U_c'}\right)g_2 = -\left(6g_0^2 - \frac{U-c}{U_c'}\right)(g_1'+g_1^2) - 12g_0g_0'g_1 - 4g_0g_0'' - 3g_0'^2 + 2\alpha^2g_0^2 - \alpha^2\frac{U-c}{U_c'} - \frac{U''}{U_c'}. \tag{5.6}$$

From equation (5.4) we see that *either* $g_0^2 = 0$ *or* $g_0^2 = (U-c)/U_c'$. If $g_0 = 0$, then equation (5.5) vanishes identically; equation (5.6), however, becomes

$$\frac{U-c}{U_c'}(g_1'+g_1^2-\alpha^2) - \frac{U''}{U_c'} = 0, \tag{5.7}$$

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which is simply a first-order nonlinear equation equivalent to Rayleigh's stability equation and we thus recover the usual inviscid solutions. Otherwise we have

$$\left.\begin{aligned} g_0(y) &= \mp\left(\frac{U-c}{U_c'}\right)^{\frac{1}{2}}, \quad g_1(y) = -\frac{5}{4}\frac{U'}{U-c} \\ \text{and} \qquad g_2(y) &= \pm\left(\frac{U_c'}{U-c}\right)^{\frac{1}{2}}\left\{\frac{101}{32}\left(\frac{U'}{U-c}\right)^2-\frac{13}{8}\frac{U''}{U-c}-\tfrac{1}{2}\alpha^2\right\}, \end{aligned}\right\} \tag{5.8}$$

where the signs of g_0 and g_2 are ordered. The branch of the square roots is fixed, for convenience, by placing a branch cut along the Stokes line

$$\tfrac{2}{3}\mathrm{ph}\int_{y_c}^{y}\{(U-c)/U_c'\}^{\frac{1}{2}}\,\mathrm{d}y = \tfrac{1}{2}\pi;$$

near the critical point, the branch cut approaches the line $\mathrm{ph}\,(y-y_c) = \frac{1}{2}\pi$.

On substituting these results into equation (5.1) we obtain two solutions which can be written in the form

$$\bar{\phi}_3(y) = \tfrac{1}{2}\pi^{-\frac{1}{2}}\epsilon^{\frac{5}{4}}\left(\frac{U-c}{U_c'}\right)^{-\frac{5}{4}}\exp\{-\epsilon^{-\frac{3}{2}}Q(y)\}\{1-\epsilon^{\frac{3}{2}}G_2(y)+O(\epsilon^3)\} \tag{5.9}$$

and

$$\bar{\phi}_4(y) = \mathrm{i}\tfrac{1}{2}\pi^{-\frac{1}{2}}\epsilon^{\frac{5}{4}}\left(\frac{U-c}{U_c'}\right)^{-\frac{5}{4}}\exp\{+\epsilon^{-\frac{3}{2}}Q(y)\}\{1+\epsilon^{\frac{3}{2}}G_2(y)+O(\epsilon^3)\}, \tag{5.10}$$

where

$$Q(y) = \int_{y_c}^{y}\left(\frac{U-c}{U_c'}\right)^{\frac{1}{2}}\mathrm{d}y \tag{5.11}$$

and

$$G_2(y) = \tfrac{101}{48}(y-y_c)^{-\frac{3}{2}}+\frac{95}{64}\frac{U_c''}{U_c'}(y-y_c)^{-\frac{1}{2}}-\int_{y_c}^{y}\left\{\left(\frac{U_c'}{U-c}\right)^{\frac{1}{2}}\left[\frac{101}{32}\left(\frac{U'}{U-c}\right)^2-\frac{13}{8}\frac{U''}{U-c}-\tfrac{1}{2}\alpha^2\right]\right.$$
$$\left.-\left(\frac{1}{y-y_c}\right)^{\frac{1}{2}}\left[\frac{101}{32}\left(\frac{1}{y-y_c}\right)^2+\frac{95}{128}\frac{U_c''}{U_c'}\frac{1}{y-y_c}\right]\right\}\mathrm{d}y. \tag{5.12}$$

In writing the solutions in this form we have, without loss of generality, fixed the normalization in a way that would appear to be particularly convenient. Thus, following Eagles (1969), we have defined $G_2(y)$ so that its expansion about $y = y_c$ contains no constant term. The remaining factors and signs in equations (5.9) and (5.10) have been chosen primarily to emphasize the close relation between these outer expansions and the familiar inner expansions of Airy function type.

The W.K.B. approximations to the solutions of equation (2.4) can easily be obtained by the same method. On fixing the normalization in the same way, we have

$$\bar{\chi}_3(\eta) = \tfrac{1}{2}\pi^{-\frac{1}{2}}\epsilon^{\frac{5}{4}}\eta'^{-1}\eta^{-\frac{5}{4}}\exp\left(-\tfrac{2}{3}\epsilon^{-\frac{3}{2}}\eta^{\frac{3}{2}}\right)\{1-\epsilon^{\frac{3}{2}}H_2(\eta)+O(\epsilon^3)\} \tag{5.13}$$

and

$$\bar{\chi}_4(\eta) = \mathrm{i}\tfrac{1}{2}\pi^{-\frac{1}{2}}\epsilon^{\frac{5}{4}}\eta'^{-1}\eta^{-\frac{5}{4}}\exp\left(+\tfrac{2}{3}\epsilon^{-\frac{3}{2}}\eta^{\frac{3}{2}}\right)\{1+\epsilon^{\frac{3}{2}}H_2(\eta)+O(\epsilon^3)\}, \tag{5.14}$$

where

$$H_2(\eta) = \tfrac{101}{48}\eta^{-\frac{3}{2}}+\tfrac{9}{5}(U_c''/U_c')\,\eta^{-\frac{1}{2}}-\int_0^{\eta}\{\tfrac{9}{2}(\gamma-\gamma_0)\eta^{-1}-\tfrac{13}{4}\gamma'+\tfrac{23}{8}\gamma^2-\tfrac{1}{2}\alpha^2\eta'^{-2}\}\eta^{-\frac{1}{2}}\,\mathrm{d}\eta \tag{5.15}$$

and the branch of the roots is again fixed by placing a branch cut along the ray $\mathrm{ph}\,\eta = \frac{1}{2}\pi$. It is easily verified by a direct calculation that $G_2(y) = H_2(\eta)$ and hence, at least to second order, $\bar{\phi}_3(y) = \bar{\chi}_3(\eta)$ and $\bar{\phi}_4(y) = \bar{\chi}_4(\eta)$.

In discussing the domains of validity of these solutions the Stokes and anti-Stokes lines, which are associated with the exponential factors in the solutions, play a particularly important role. These lines are defined by the conditions $\mathrm{Im}\,(\epsilon^{-\frac{3}{2}}\eta^{\frac{3}{2}}) = 0$ and $\mathrm{Re}\,(\epsilon^{-\frac{3}{2}}\eta^{\frac{3}{2}}) = 0$ respectively (see

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figure 1). Consider first the solution $\overline{\chi}_3(\eta)$ and let the branch cut be shifted temporarily to the anti-Stokes line $\mathrm{ph}\,\eta = \frac{5}{6}\pi$. Then $\overline{\chi}_3(\eta)$ is recessive in $\boldsymbol{S}_1$ but dominant in $\boldsymbol{S}_2$ and $\boldsymbol{S}_3$, and its domain of validity is normally taken to be the open sector of angle 2π excluding the ray $\mathrm{ph}\,\eta = \frac{5}{6}\pi$. The same conclusion can be reached without moving the branch cut by applying phase-integral theory (cf. Heading 1962), but it is then necessary to allow for a change in the form but not the value of $\overline{\chi}_3(\eta)$ on crossing the branch cut. In parts of this sector, however, it would appear that

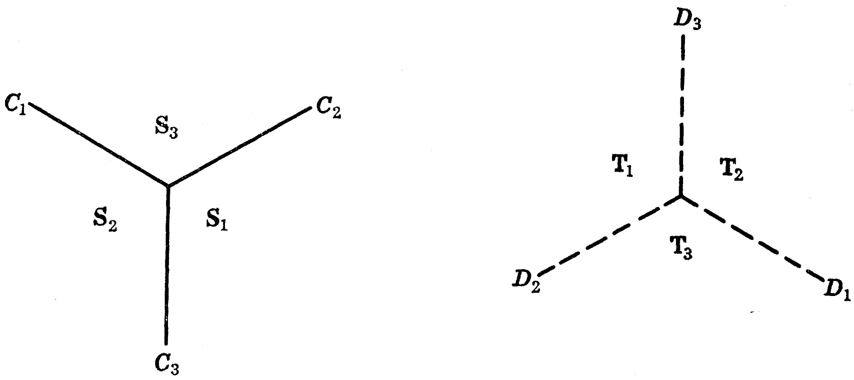

FIGURE 1. The anti-Stokes lines (left) and the Stokes lines (right) in the η-plane.

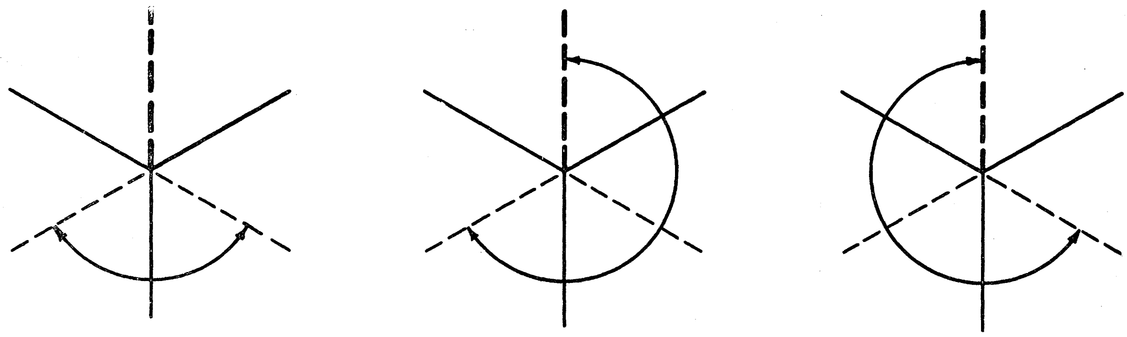

FIGURE 2. The restricted sectors of validity of $\chi_2^{(0)}(\eta)$ (left), $\overline{\chi}_3(\eta)$ (centre) and $\overline{\chi}_4(\eta)$ (right) in which they are complete asymptotic expansions. The heavy dashed lines denote branch cuts.

$\overline{\chi}_3(\eta)$ is not a complete asymptotic expansion and we must therefore restrict its domain of validity appropriately. A rigorous justification of this step cannot be given at the present time since that would require a theory of error bounds and such a theory has not yet been developed for equations of the Orr–Sommerfeld type. Olver's (1961) work on error bounds for W.K.B. approximations to the solutions of second-order differential equations strongly suggests, however, that we restrict the domain of validity of $\overline{\chi}_3(\eta)$ to the smaller sector $-\frac{5}{6}\pi < \mathrm{ph}\,\eta < \frac{1}{2}\pi$ the boundaries of which are Stokes lines. Similar considerations suggest a restriction of $\overline{\chi}_4(\eta)$ to the sector $-\frac{3}{2}\pi < \mathrm{ph}\,\eta < -\frac{1}{6}\pi$. The corresponding domains of validity of $\overline{\phi}_3(y)$ and $\overline{\phi}_4(y)$ in the y-plane then follow immediately from the foregoing discussion by simply replacing $\mathrm{ph}\,\eta$ by

$$\tfrac{2}{3}\,\mathrm{ph}\int_{y_c}^{y}\{(U-c)/U_c'\}^{\frac{1}{2}}\,\mathrm{d}y.$$

These results, as well as those obtained in §4 for $\chi_2^{(0)}(\eta)$, are summarized in figure 2.

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6. Determination of the coefficients in the expansion by matching

In § 3 we derived the differential equations satisfied by the coefficients in the expansion (3.4) but, because of the rather complicated structure of those equations, a complete discussion was given only for the first four coefficients A_0, B_0, C_0 and D_0. Furthermore, the determination of B_1, which is required in the present theory, could not easily be done by that method. An alternative approach to the problem of determining the coefficients in the expansion has been suggested by Lin (1957*a*) which not only recovers the results obtained in § 3 but also easily leads to an explicit determination of B_1. In this method, outer expansions to the solutions of the comparison equation are used in equation (3.4) and the resulting expressions are then matched, in appropriate sectors of the η-plane, to the inviscid approximations of § 4 and the W.K.B. approximations of § 5.

In discussing this matching method it is convenient to let $\tilde{\chi}_k$ ($k = 1, 2, 3, 4$) denote the uniform approximations obtained from the expansion (3.4) by using the exact solutions u_k of the comparison equation (cf. equations (A 24)). These uniform approximations are, of course, valid throughout the whole of D_η but, since we have only integral representations for the exact solutions of the comparison equation, they are of a much too complicated form to be used in actual calculations. Once the necessary coefficients in the expansion (3.4) have been found, either by the differential equation method discussed in § 3 or by the matching method to be discussed in this section, we can use the outer expansions of the u_k to obtain the corresponding outer expansions of the $\tilde{\chi}_k$ from which the Stokes multipliers can then be immediately obtained.

To simplify the present discussion we will use the fact that $B_0 + \eta D_0 = 0$. The outer expansion of $\tilde{\chi}_1$ must clearly be of the form $\tilde{\chi}_1 = \tilde{\chi}_1^{(0)} + O(\epsilon^3)$ for all values of ph η and hence

$$\tilde{\chi}_1^{(0)} = A_0 u_1^{(0)} - \eta D_0 u_1^{(0)\prime}.$$

Furthermore, since $\tilde{\chi}_1^{(0)}$ is well-balanced, it can differ from $\chi_1^{(0)}$ by at most a multiplicative factor. Similarly, the outer expansion of $\tilde{\chi}_2$ in $\mathbf{T}_3$ must also be of the form $\tilde{\chi}_2 = \tilde{\chi}_2^{(0)} + O(\epsilon^3)$ so that $\tilde{\chi}_2^{(0)} = A_0 u_2^{(0)} - \eta D_0 u_2^{(0)\prime}$ and clearly it must be possible to express $\tilde{\chi}_2^{(0)}$ as a linear combination of $\chi_1^{(0)}$ and $\chi_2^{(0)}$. Thus, A_0 and D_0 are determined by the equations

$$A_0 u_1^{(0)} - \eta D_0 u_1^{(0)\prime} = c_1 \chi_1^{(0)} \quad (\eta \in \mathbf{I}), \tag{6.1}$$

and

$$A_0 u_2^{(0)} - \eta D_0 u_2^{(0)\prime} = c_0 \chi_1^{(0)} + c_2 \chi_2^{(0)} \quad (\eta \in \mathbf{T}_3), \tag{6.2}$$

where the constants c_0, c_1 and c_2 must be chosen so as to make A_0 and D_0 analytic at $\eta = 0$. Since $W(u_1^{(0)}, u_2^{(0)}) = -1$, we have

$$A_0 = (c_0 \chi_1^{(0)} + c_2 \chi_2^{(0)})\, u_1^{(0)\prime} - c_1 \chi_1^{(0)} u_2^{(0)\prime} \tag{6.3}$$

and

$$\eta D_0 = (c_0 \chi_1^{(0)} + c_2 \chi_2^{(0)})\, u_1^{(0)} - c_1 \chi_1^{(0)} u_2^{(0)}. \tag{6.4}$$

The condition that A_0 and D_0 be analytic at $\eta = 0$ requires only that $c_1 = c_2$ and it can easily be verified that these expressions for A_0 and D_0 then satisfy the differential equations (3.18) and (3.19) for arbitrary values of c_0 and c_1. As in § 3, we can, without loss of generality, choose $c_1 = 1$; this is also perhaps the most natural choice in view of the way in which $u_1^{(0)}$ and $\chi_1^{(0)}$ have been normalized. From equations (6.3) and (6.4) we then have the initial values

$$A_0(0) = 1, \quad A_0'(0) = c_0, \quad D_0(0) = 0 \quad \text{and} \quad D_0'(0) = c_0 - \tfrac{1}{5}(U_c''/U_c'), \tag{6.5}$$

which thus define a one parameter family of solutions of the differential equations (3.18) and (3.19). Since $u_2^{(0)}$ and $\chi_2^{(0)}$ have been defined so that their regular parts contain no multiple of $u_1^{(0)}$

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and $\chi_1^{(0)}$ respectively, it is again natural and convenient, as in §3, to choose $c_0 = 0$. With these values for c_0, c_1 and c_2 we can rewrite equations (6.3) and (6.4) in the forms

$$A_0(\eta) = Q_2(\eta)u_1^{(0)\prime}(\eta) - \eta Q_1(\eta)\,\{\text{regular part of } u_2^{(0)\prime}(\eta)\} \tag{6.6}$$

and

$$\eta D_0(\eta) = Q_2(\eta)u_1^{(0)}(\eta) - \eta Q_1(\eta)\,\{\text{regular part of } u_2^{(0)}(\eta)\}, \tag{6.7}$$

where the regular part of $u_2^{(0)}(\eta)$ can be found from equation (A 28). These expressions show explicitly that A_0 and D_0 are both analytic at $\eta = 0$. Furthermore, the power series for A_0 and D_0 obtained from them are in complete agreement with the results given in §3.

Thus we have two quite distinct methods of determining A_0 and D_0: one based on the solutions of two *coupled* second-order differential equations and the other based on the solutions of two *singular* second-order differential equations. It may be noted, however, that given the two coupled equations, as in §3, but without any knowledge of the matching method just described, it is by no means obvious how one could obtain the required solutions in the form of products of the solutions of two singular but uncoupled equations. Although it is unnecessary in the present theory to actually compute A_0 and D_0, and we have therefore not studied the computational problem in any great detail, it would appear that the two methods involve comparable amounts of work.

Consider next the determination of C_0 and B_1 by the matching method. For this purpose it is convenient to note first the inner expansion of $\tilde{\chi}_3$ which follows from equation (A 9) in the form

$$\begin{aligned}\tilde{\chi}_3(\eta) = \epsilon^{-\alpha}\{(A_0-D_0)\,A_1(\zeta,1) + \epsilon[\beta_0(A_0-2D_0)\,A_1(\zeta,2) + C_0A_1(\zeta,-1)]\\ + \epsilon^2[\tfrac{1}{2}\beta_0^2\,(A_0-3D_0)\,A_1(\zeta,3) + (\beta_0C_0-B_1)\,A_1(\zeta,0)] + O(\epsilon^3)\}.\end{aligned} \tag{6.8}$$

On re-expansion of this result in the sector $|\mathrm{ph}\,\zeta| < \frac{2}{3}\pi$ or by a direct calculation using equation (A 12), we obtain the outer expansion

$$\begin{aligned}\tilde{\chi}_3(\eta) = \tfrac{1}{2}\pi^{-\frac{1}{2}}\,\epsilon^{-\alpha+\frac{5}{4}}\,\eta^{-\frac{1}{4}}\exp\left(-\tfrac{2}{3}\epsilon^{-\frac{3}{2}}\eta^{\frac{3}{2}}\right)[\![A_0-D_0+\eta C_0-\{\tfrac{101}{48}(A_0-D_0)\eta^{-\frac{3}{2}}\\ +[\tfrac{5}{48}C_0-\beta_0(A_0-2D_0)]\eta^{-\frac{1}{2}}-(\beta_0C_0-B_1)\eta^{\frac{1}{2}}\}\epsilon^{\frac{3}{2}}+O(\epsilon^3)]\!]\quad(\eta\in\mathbf{T}_2\cup\mathbf{T}_3).\end{aligned} \tag{6.9}$$

It is particularly interesting to note the manner in which C_0 and B_1 appear in these expressions. For a first approximation of the outer type it is clearly necessary to determine C_0 and for a second approximation one must also determine B_1, though this was certainly not obvious from the discussion of §3. The matching principle then asserts that, in the sector $\mathbf{T}_2 \cup \mathbf{T}_3$, the outer expansion of $\tilde{\chi}_3$ must be a multiple of the W.K.B. approximation (5.13). More precisely, we have $\tilde{\chi}_3 = c_3\bar{\chi}_3$, where c_3 must, in general, have an asymptotic expansion of the form

$$c_3(\epsilon) = \epsilon^{-\alpha}\sum_{n=0}^{\infty}(-1)^n c_3^{(n)}\,\epsilon^{\frac{3}{2}n}. \tag{6.10}$$

To first order the matching implies that

$$A_0 - D_0 + \eta C_0 = c_3^{(0)}\eta'^{-1}, \tag{6.11}$$

and the condition that C_0 be analytic at $\eta = 0$ then requires that $c_3^{(0)} = 1$. Thus we have

$$C_0 = \frac{1}{\eta}\left(\frac{1}{\eta'} - A_0 + D_0\right) \tag{6.12}$$

in agreement with equation (3.24). To second order the matching implies that

$$\{\tfrac{101}{48}(A_0-D_0)\eta^{-\frac{3}{2}} + [\tfrac{5}{48}C_0-\beta_0(A_0-2D_0)]\eta^{-\frac{1}{2}} - (\beta_0C_0-B_1)\eta^{\frac{1}{2}}\} = (1/\eta')\,\{H_2(\eta) - c_3^{(1)}\}, \tag{6.13}$$

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and the condition that B_1 be analytic at $\eta = 0$ then requires that $c_3^{(1)} = 0$. This value of $c_3^{(1)}$ is essentially a consequence of the way in which the W.K.B. approximations have been normalized and, more specifically, of the fact that there is no constant term in the expansion of $H_2(\eta)$ about $\eta = 0$ [cf. also equation (A 12)]. After some simplification we obtain

$$B_1 = -2(A_0 - D_0 - \eta'^{-1})\eta^{-2} - \beta_0(D_0 + \tfrac{4}{5}\eta'^{-1})\eta^{-1} - \eta'^{-1}\eta^{-\frac{1}{2}}\int_0^{\eta}\{\tfrac{9}{2}(\gamma - \gamma_0)\eta^{-1} - \tfrac{13}{4}\gamma' + \tfrac{23}{8}\gamma^2 - \tfrac{1}{2}\alpha^2\eta'^{-2}\}\eta^{-\frac{1}{2}}\,\mathrm{d}\eta, \quad (6.14)$$

from which we also have

$$B_1(0) = -\frac{U_c'''}{U_c'} + \frac{17}{30}\frac{U_c''^2}{U_c'^2} + \tfrac{2}{3}\alpha^2 \quad (6.15)$$

in agreement with equation (3.33). This expression for B_1 is somewhat complicated and it is perhaps not altogether surprising that we were unable to obtain it in a simple way from the differential equations derived in § 3.

In this method of determining B_1, no reference has thus far been made to the value of α_1. We can, however, now give a somewhat simpler method of finding its value. From the expansion (3.4) we can easily derive the relation

$$\chi_1^{(1)}(0) = A_0(0)\,u_1^{(1)}(0) + B_1(0)\,u_1^{(0)\prime}(0) + C_0(0)\,u_1^{(0)\prime\prime}(0), \quad (6.16)$$

in which all quantities are known except $u_1^{(1)}(0)$. Now $u_1^{(1)}$ satisfies the inhomogeneous equation

$$\eta u_1^{(1)\prime\prime} + \beta_0 u_1^{(1)} = u_1^{(0)\mathrm{iv}} - \alpha_1 u_1^{(0)\prime} - \beta_1 u_1^{(0)}, \quad (6.17)$$

and since $u_1^{(1)}$ is analytic at $\eta = 0$ we find, on setting $\eta = 0$ in this equation, that

$$u_1^{(1)}(0) = -\tfrac{1}{6}\beta_0^2 - \alpha_1\beta_0^{-1}. \quad (6.18)$$

A short calculation then confirms the value of α_1 given by equation (3.31).

The matching principle further asserts that, in the sector $\mathbf{T}_1 \cup \mathbf{T}_3$, the outer expansion of $\tilde{\chi}_4$ must be a multiple of the W.K.B. approximation (5.14). Thus, if we let $\tilde{\chi}_4 = c_4\bar{\chi}_4$, then a similar analysis shows that c_4 must also have an asymptotic expansion of the form

$$c_4(\epsilon) = \epsilon^{-\alpha}\sum_{n=0}^{\infty} c_3^{(n)}\,\epsilon^{\frac{3}{2}n}. \quad (6.19)$$

Having determined the outer expansion of $\tilde{\chi}_2$ in $\mathbf{T}_3$, $\tilde{\chi}_3$ in $\mathbf{T}_2 \cup \mathbf{T}_3$ and $\tilde{\chi}_4$ in $\mathbf{T}_1 \cup \mathbf{T}_3$, we must now consider the problem of determining their outer expansions in the complimentary sectors $\mathbf{T}_1 \cup \mathbf{T}_2$, $\mathbf{T}_1$ and $\mathbf{T}_2$ respectively. This is really the crucial step, as will be seen in the following section, in the determination of the Stokes multipliers.

7. The Stokes multipliers

The outer expansions of $\tilde{\chi}_2$, $\tilde{\chi}_3$ and $\tilde{\chi}_4$ which were derived in the previous section are valid in the complete sense of Olver in the sectors $\mathbf{T}_3$, $\mathbf{T}_2 \cup \mathbf{T}_3$ and $\mathbf{T}_1 \cup \mathbf{T}_3$ respectively.† On crossing the Stokes lines which bound these sectors, however, a change takes place in the form of the outer expansions in accordance with the well-known Stokes phenomenon. To obtain the outer expansions of $\tilde{\chi}_2$, $\tilde{\chi}_3$ and $\tilde{\chi}_4$ in the sectors $\mathbf{T}_1 \cup \mathbf{T}_2$, $\mathbf{T}_1$ and $\mathbf{T}_2$ respectively we must now make explicit use of the connexion formulae for the solutions of the comparison equation.

† They remain valid, of course, in the ordinary Poincaré sense in the extended sectors $\mathbf{S}_1 \cup \mathbf{S}_2$, $\mathbf{I} - \boldsymbol{C}_1$ and $\mathbf{I} - \boldsymbol{C}_2$ respectively but the errors associated with the expansions may be expected to become arbitrarily large near the boundaries of these extended sectors.

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Consider first the behaviour of $\tilde{\chi}_2$ whose outer expansion is purely balanced in $\mathbf{T}_3$. On crossing the Stokes line $\boldsymbol{D}_1$ from $\mathbf{T}_3$ to $\mathbf{T}_2$, the outer expansion of $\tilde{\chi}_2$ must pick up a multiple of $\overline{\chi}_3$ which is maximally recessive on $\boldsymbol{D}_1$; similarly, on crossing $\boldsymbol{D}_2$ from $\mathbf{T}_3$ to $\mathbf{T}_1$ it must pick up a multiple of $\overline{\chi}_4$ which is maximally recessive on $\boldsymbol{D}_2$. Thus we can write

$$\tilde{\chi}_2 = \chi_2^{(0)} + O(\epsilon^3) + \begin{Bmatrix} s_4(\epsilon)\,\overline{\chi}_4 & (\eta\in\mathbf{T}_1) \\ s_3(\epsilon)\,\overline{\chi}_3 & (\eta\in\mathbf{T}_2) \\ 0 & (\eta\in\mathbf{T}_3) \end{Bmatrix}, \tag{7.1}$$

where $\operatorname{ph}\eta$ is restricted to the range $-\frac{3}{2}\pi < \operatorname{ph}\eta < \frac{1}{2}\pi$ and the Stokes multipliers s_3 and s_4 must, in general, be allowed to depend on ϵ asymptotically. The asymptotic expansions of s_3 and s_4 depend, of course, on the way in which $\chi_2^{(0)}$, $\overline{\chi}_3$ and $\overline{\chi}_4$ are normalized. Once they have been determined for a given normalization, however, the corresponding expansions for any other normalization can easily be obtained from them. Thus, from equation (A 26), we immediately obtain

$$s_3(\epsilon) = -2\pi i\beta_0\,\epsilon c_3(\epsilon) \quad\text{and}\quad s_4(\epsilon) = 2\pi i\beta_0\,\epsilon c_4(\epsilon) \tag{7.2}$$

or, to first order,

$$s_3(\epsilon) = -2\pi i\beta_0\,\epsilon\{1+O(\epsilon^3\ln\epsilon)\} \quad\text{and}\quad s_4(\epsilon) = 2\pi i\beta_0\,\epsilon\{1+O(\epsilon^3\ln\epsilon)\}. \tag{7.3}$$

Consider next the behaviour of $\tilde{\chi}_3$ whose outer expansion is recessive in $\mathbf{S}_1$ and purely dominant in $(\mathbf{T}_2\cup\mathbf{T}_3)-\mathbf{S}_1$. On crossing the Stokes line $\boldsymbol{D}_2$ from $\mathbf{T}_3$ to $\mathbf{T}_1$, the outer expansion of $\tilde{\chi}_3$ must pick up not only a multiple of $\overline{\chi}_4$ which is maximally recessive on $\boldsymbol{D}_2$ but also multiplies of the outer expansion of $\tilde{\chi}_1$ and the balanced part of the outer expansion of $\tilde{\chi}_2$. Thus, from equation (A 21), we have

$$\tilde{\chi}_3 = c_3(\epsilon)\,\overline{\chi}_3 - c_4(\epsilon)\,\overline{\chi}_4 + s_1(\epsilon)\,\{\chi_1^{(0)}+O(\epsilon^3)\} + s_2(\epsilon)\,\{\chi_2^{(0)}+O(\epsilon^3)\} \quad (\eta\in\mathbf{T}_1), \tag{7.4}$$

where

$$s_1(\epsilon) = -\epsilon^{-1}+O(\epsilon^2) \quad\text{and}\quad s_2(\epsilon) = O(\epsilon^2). \tag{7.5}$$

Similarly, from equation (A 22), we have

$$\tilde{\chi}_4 = c_4(\epsilon)\,\overline{\chi}_4 - c_3(\epsilon)\,\overline{\chi}_3 + s_1(\epsilon)\,\{\chi_1^{(0)}+O(\epsilon^3)\} + s_2(\epsilon)\,\{\chi_2^{(0)}+O(\epsilon^3)\} \quad (\eta\in\mathbf{T}_2). \tag{7.6}$$

For purposes of deriving the characteristic equation and to simplify the subsequent calculations, it is desirable to transform these results back to the y-plane. Thus, if we let

$$\{\phi_1, \phi_2, \phi_3, \phi_4\} = \eta'^{-\frac{3}{2}}\{\tilde{\chi}_1, \tilde{\chi}_2 + \tfrac{3}{10}(U_c''/U_c')\,\tilde{\chi}_1, \tilde{\chi}_3, \tilde{\chi}_4\} \tag{7.7}$$

then the corresponding outer expansions of the ϕ's can immediately be obtained in the form

$$\phi_1 = \phi_1^{(0)} + O(\epsilon^3) \quad (y\in\mathbf{I}), \tag{7.8}$$

$$\phi_2 = \phi_2^{(0)} + O(\epsilon^3) + \begin{Bmatrix} s_4(\epsilon)\,\overline{\phi}_4 & (y\in\mathbf{T}_1) \\ s_3(\epsilon)\,\overline{\phi}_3 & (y\in\mathbf{T}_2) \\ 0 & (y\in\mathbf{T}_3) \end{Bmatrix}, \tag{7.9}$$

$$\phi_3 = c_3(\epsilon)\,\overline{\phi}_3 + \begin{Bmatrix} -c_4(\epsilon)\,\overline{\phi}_4 + s_1(\epsilon)\,\phi_1^{(0)} + O(\epsilon^2) & (y\in\mathbf{T}_1) \\ 0 & (y\in\mathbf{T}_2\cup\mathbf{T}_3) \end{Bmatrix}, \tag{7.10}$$

and

$$\phi_4 = c_4(\epsilon)\,\overline{\phi}_4 + \begin{Bmatrix} -c_3(\epsilon)\,\overline{\phi}_3 + s_1(\epsilon)\,\phi_1^{(0)} + O(\epsilon^2) & (y\in\mathbf{T}_2) \\ 0 & (y\in\mathbf{T}_1\cup\mathbf{T}_3) \end{Bmatrix}. \tag{7.11}$$

In the y-plane, of course, the Stokes and anti-Stokes lines are no longer straight; near the critical point y_c, however, they have the same arrangement as shown in figure 1.

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8. The characteristic equation

We now wish to derive an approximation to the characteristic equation based on the outer expansions (7.8) to (7.11). For symmetrical flow in a channel we suppose that $U(y)$ is monotone increasing on the interval $y_1 \leqslant y \leqslant y_2$ and for an even solution we have the usual boundary conditions

$$\phi = \phi' = 0 \quad \text{at} \quad y = y_1 \qquad \text{and} \qquad \phi' = \phi''' = 0 \quad \text{at} \quad y = y_2. \tag{8.1}$$

The solutions ϕ_3 and ϕ_4 are dominant at y_1 and y_2 respectively. But if y_c is substantially closer to y_1 than to y_2, as it is in most problems, then $|\phi_4(y_2)| \gg |\phi_3(y_1)|$ and with a very small error we can therefore reject ϕ_4. This approximation merely reflects the fact that, except near y_1 and y_c, the solution has a largely inviscid character. Thus we let

$$\phi = A\phi_1 + \phi_2 + C\phi_3, \tag{8.2}$$

where the coefficient of ϕ_2 has been chosen to be unity to fix the normalization.

In applying the boundary conditions at $y = y_2$ we can, consistent with the rejection of ϕ_4, also neglect ϕ_3 and its derivatives since they are very much smaller than all terms retained in the subsequent analysis. Thus, near y_2, we have

$$\phi = A\phi_1^{(0)} + \phi_2^{(0)} + O(\epsilon^3). \tag{8.3}$$

If, as usual, we now let $\Phi = A\phi_1^{(0)} + \phi_2^{(0)}$ then A is determined by the condition $\Phi'(y_2) = 0$, i.e. $A = -\phi_2^{(0)\prime}(y_2)/\phi_1^{(0)\prime}(y_2)$. For small values of α and c we note that

$$\left.\begin{aligned} A(\alpha, c) &= U_1'^2 I_2^{-1} \alpha^{-2}\{1 + O(\alpha^2, c)\}, \\ \text{where} \qquad I_2 &= \int_{y_1}^{y_2} U^2 \,\mathrm{d}y. \end{aligned}\right\} \tag{8.4}$$

In this approximation, the boundary condition $\phi'''(y_2) = 0$ is automatically satisfied because of the symmetry of $U(y)$.

In the sector $\mathbf{T}_1$, which contains y_1, we then have from equations (7.9) and (7.10)

$$\phi = \Phi + O(\epsilon^3) + s_4(\epsilon)\,\overline{\phi}_4 + C\{c_3(\epsilon)\,\overline{\phi}_3 - c_4(\epsilon)\,\overline{\phi}_4 + s_1(\epsilon)\,\phi_1^{(0)} + O(\epsilon^2)\} \tag{8.5}$$

and the required approximation to the characteristic equation can therefore be written in the form

$$\frac{\Phi'(y_1) + O(\epsilon^3) + s_4(\epsilon)\,\overline{\phi}_4'(y_1)}{\Phi(y_1) + O(\epsilon^3) + s_4(\epsilon)\,\overline{\phi}_4(y_1)} = \frac{c_3(\epsilon)\,\overline{\phi}_3'(y_1) - c_4(\epsilon)\,\overline{\phi}_4'(y_1) + s_1(\epsilon)\,\phi_1^{(0)\prime}(y_1) + O(\epsilon^2)}{c_3(\epsilon)\,\overline{\phi}_3(y_1) - c_4(\epsilon)\,\overline{\phi}_4(y_1) + s_1(\epsilon)\,\phi_1^{(0)}(y_1) + O(\epsilon^2)}. \tag{8.6}$$

Although this is a convenient form in which to write the characteristic equation, it does not exhibit in any very explicit way the relative orders of magnitude of the terms which appear in it. A more useful form for the present purposes can be obtained, however, by first rewriting equation (8.6) as the difference of two products. On substituting for $\overline{\phi}_3$ and $\overline{\phi}_4$ from equations (5.9) and (5.10) and neglecting terms of order $\epsilon^3 \ln \epsilon$ or ϵ^3 compared to unity, we then obtain

$$\Delta_1(y_1)\exp\{-\epsilon^{-\frac{3}{2}}Q(y_1)\} + \Delta_2(y_1) + \Delta_3(y_1)\exp\{\epsilon^{-\frac{3}{2}}Q(y_1)\} = 0, \tag{8.7}$$

where

$$\Delta_1(y) = 1 + \epsilon^{\frac{3}{2}}\left[\left(\frac{U-c}{U_c'}\right)^{-\frac{1}{2}}\left(\frac{\Phi'}{\Phi} + \frac{5}{4}\frac{U'}{U-c}\right) - G_2(y)\right] + O(\epsilon^3 \ln \epsilon), \tag{8.8}$$

$$\Delta_2(y) = 2\pi^{\frac{1}{2}}\epsilon^{-\frac{3}{4}}\left(\frac{U-c}{U_c'}\right)^{\frac{3}{4}}\frac{1}{\Phi}\{1 + O(\epsilon^3)\}, \tag{8.9}$$

$$\Delta_3(y) = \mathrm{i}\frac{\Psi}{\Phi}\left\{1 - \epsilon^{\frac{3}{2}}\left[\left(\frac{U-c}{U_c'}\right)^{-\frac{1}{2}}\left(\frac{\Psi'}{\Psi} + \frac{5}{4}\frac{U'}{U-c}\right) - G_2(y)\right] + O(\epsilon^3 \ln \epsilon)\right\}, \tag{8.10}$$

and

$$\Psi = \Phi - 2\pi\mathrm{i}\beta_0\phi_1^{(0)}. \tag{8.11}$$

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In the derivation of this form of the characteristic equation there is one important point which should be especially noted. Consider, for example, the dominant term in equation (8.7) which, to within a multiplicative factor, is obtained from $c_3(\epsilon)\{\Phi'(y)\,\overline{\phi}_3(y_1)-\Phi(y_1)\,\overline{\phi}_3'(y_1)\}$. In this expression Φ and Φ' are both of order unity but $\overline{\phi}_3'$ is larger than $\overline{\phi}_3$ by a factor of order $\epsilon^{-\frac{1}{3}}$. Accordingly, to obtain a consistent approximation to $\Delta_1(y_1)$ it is necessary to retain two terms in $\overline{\phi}_3'$ but only one term in $\overline{\phi}_3$. Similar remarks also apply to the balanced and recessive terms in equation (8.7). This manner of ordering terms in equation (8.7) has some important implications which will be discussed later in this section.

Equation (8.7) can also be written in a somewhat more explicit form which is particularly useful for computational purposes. Thus, if we introduce the usual variables

$$Z = c(\alpha R/U_1'^2)^{\frac{1}{3}} \quad \text{and} \quad \hat{z} = (\alpha R)^{\frac{1}{3}}\left\{\frac{3}{2}\int_{y_1}^{y_c}|U-c|^{\frac{1}{2}}\,\mathrm{d}y\right\}^{\frac{2}{3}}, \tag{8.12}$$

where Z and $\hat{z}$ are real and positive in the case of neutral stability, then equation (8.7) becomes

$$\Delta_1(y_1)\exp\left(\tfrac{2}{3}\hat{z}^{\frac{3}{2}}\,\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}\right)+\Delta_2(y_1)+\Delta_3(y_1)\exp\left(-\tfrac{2}{3}\hat{z}^{\frac{3}{2}}\,\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}\right) = 0, \tag{8.13}$$

where

$$\Delta_1(y_1) = 1+\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}\,Z^{-\frac{3}{2}}\left[\frac{c}{U_1'}\frac{\Phi'(y_1)}{\Phi(y_1)}-\frac{5}{4}+\mathrm{i}\,\frac{c^{\frac{3}{2}}}{U_c'^{\frac{1}{2}}U_1'}\,G_2(y_1)\right]+O(Z^{-3}\ln Z), \tag{8.14}$$

$$\Delta_2(y_1) = 2\pi^{\frac{1}{2}}\,\mathrm{e}^{-\frac{5}{8}\pi\mathrm{i}}\left(\frac{U_1'}{U_c'}\right)^{\frac{1}{2}}\frac{1}{\Phi(y_1)}\,Z^{\frac{3}{4}}\{1+O(Z^{-3})\}, \tag{8.15}$$

and

$$\Delta_3(y_1) = \mathrm{i}\,\frac{\Psi(y_1)}{\Phi(y_1)}\left\{1-\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}\,Z^{-\frac{3}{2}}\left[\frac{c}{U_1'}\frac{\Psi'(y_1)}{\Psi(y_1)}-\frac{5}{4}+\mathrm{i}\,\frac{c^{\frac{3}{2}}}{U_c'^{\frac{1}{2}}\,U_1'}\,G_2(y_1)\right]+O(Z^{-3}\ln Z)\right\}. \tag{8.16}$$

It is also of interest to consider the limiting form of equation (8.13) as α and c tend to zero; for it is this limit which determines the asymptotes to the upper and lower branches of the neutral curve, at least for flows without an inflexion point. In this limit we have

$$\left.\begin{aligned}
\phi_1^{(0)}(y_1) &= -(c/U_1')\{1+O(c)\},\\
\phi_2^{(0)}(y_1) &= 1+O(c\ln c)+\mathrm{i}\pi(U_1''/U_1'^2)c\{1+O(c)\},\\
\phi_1^{(0)\prime}(y_1) &= 1+O(c),\\
\phi_2^{(0)\prime}(y_1) &= (U_1''/U_1')\ln c+O(1)-\mathrm{i}\pi(U_1''/U_1')\{1+O(c)\},
\end{aligned}\right\} \tag{8.17}$$

$$G_2(y_1) = -\mathrm{i}\tfrac{101}{48}(U_1'/c)^{\frac{3}{2}}+O(c^{-\frac{1}{2}}) \quad \text{and} \quad \hat{z} = Z\{1+O(c)\}. \tag{8.18}$$

If we now let $X = (I_2/U_1')(\alpha^2/c)$, where X remains of order unity as α and c tend to zero, then after some manipulation, we find from equation (8.13) that

$$1-X-\mathrm{i}\pi(U_1''\,I_2/U_1'^3)\alpha^2X = f(Z) \quad \text{(say)}, \tag{8.19}$$

where

$$f(Z) = \frac{1-\pi^{-\frac{1}{2}}\,\mathrm{e}^{-\frac{3}{8}\pi\mathrm{i}}\,Z^{-\frac{9}{4}}\cos\left(\Xi-\frac{1}{4}\pi\right)}{1+\pi^{-\frac{1}{2}}\,\mathrm{e}^{-\frac{1}{8}\pi\mathrm{i}}\,Z^{-\frac{3}{4}}\left\{\sin\left(\Xi-\frac{1}{4}\pi\right)-\frac{41}{72}\Xi^{-1}\cos\left(\Xi-\frac{1}{4}\pi\right)\right\}} \tag{8.20}$$

and

$$\Xi \equiv \tfrac{2}{3}Z^{\frac{3}{2}}\,\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}, \tag{8.21}$$

and we have now dropped the error estimates. The right-hand side of this equation bears a remarkable but not altogether unexpected relationship to the asymptotic expansion of the well-known Tietjens function $F(Z)$ defined by (cf. Hughes & Reid 1968)

$$F(Z) = \frac{A_1(Z\,\mathrm{e}^{-\frac{5}{6}\pi\mathrm{i}},1)}{Z\,\mathrm{e}^{-\frac{5}{6}\pi\mathrm{i}}\,A_1'(Z\,\mathrm{e}^{-\frac{5}{6}\pi\mathrm{i}},1)}. \tag{8.22}$$

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In the sector $-\frac{1}{2}\pi < \mathrm{ph}\, Z < \frac{1}{6}\pi$, the asymptotic expansion of $F(Z)$ in the complete sense is given by

$$F(Z) = \frac{1-\pi^{-\frac{1}{2}}\,\mathrm{e}^{-\frac{3}{8}\pi\mathrm{i}}\,Z^{-\frac{9}{4}}\{v_1\cos(\Xi-\frac{1}{4}\pi)+v_2\sin(\Xi-\frac{1}{4}\pi)\}}{1+\pi^{-\frac{1}{2}}\,\mathrm{e}^{-\frac{1}{8}\pi\mathrm{i}}\,Z^{-\frac{3}{4}}\{u_1\sin(\Xi-\frac{1}{4}\pi)-u_2\cos(\Xi-\frac{1}{4}\pi)\}}, \tag{8.23}$$

where u_1, u_2, v_1, v_2 are asymptotic power series, the leading terms of which are given by

$$\left.\begin{aligned} u_1 &= 1+O(\Xi^{-2}), \quad u_2 = \tfrac{41}{72}\Xi^{-1}+O(\Xi^{-3}),\\ v_1 &= 1+O(\Xi^{-2}), \quad v_2 = \tfrac{101}{72}\Xi^{-1}+O(\Xi^{-3}).\end{aligned}\right\} \tag{8.24}$$

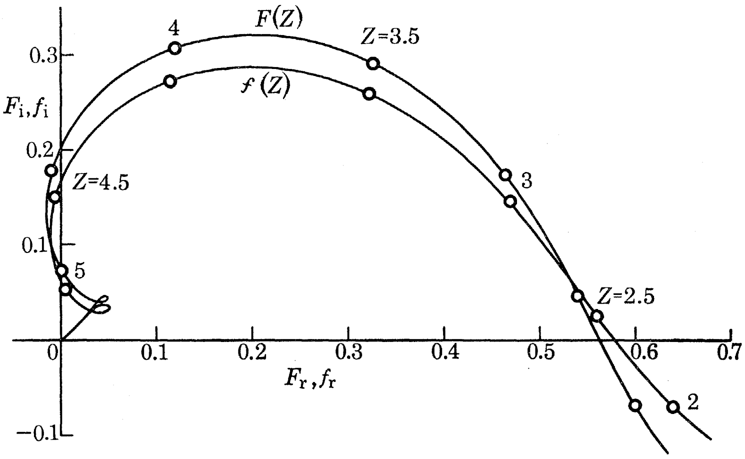

FIGURE 3. A comparison of $f(Z)$ with the Tietjens function $F(Z)$.

[Additional terms in these asymptotic power series are given by Luke (1962, pp. 137 and 139).] A comparison between $f(Z)$ and $F(Z)$ is shown in figure 3. Equation (8.19) is, of course, only relevant when the imaginary part of $f(Z)$ is small and this is seen to occur when $Z\to\infty$ or $Z\to Z_0 = 2.380$. When $Z\to\infty$,

$$X\to 1, \quad \text{i.e. } c \sim (I_2/U_1')\alpha^2, \quad \text{and} \quad -\pi(U_1''I_2/U_1'^3)\alpha^2 \sim 2^{-\frac{1}{2}}Z^{-\frac{3}{2}},$$

i.e. $R \sim \frac{1}{2}\pi^{-2}(U_1'^{11}/U_1''^2 I_2^5)\alpha^{-11}$ which is simply the asymptote to the upper branch of the neutral curve. When $Z\to Z_0$, however, $X\to 1-\mathrm{Re}\{f(Z_0)\}$, where $\mathrm{Re}\{f(Z_0)\} = 0.5797$, and along the lower branch of the neutral curve we have

$$c \sim 2.379\,(I_2|U_1')\alpha^2 \quad \text{and} \quad R \sim 1.001\,(U_1'^5/I_2^3)\alpha^{-7}. \tag{8.25}$$

For comparison we may note the corresponding numerical values associated with the Tietjens function: $Z_0 = 2.297$ and $\mathrm{Re}\{F(Z_0)\} = 0.5645$. Furthermore, the exact behaviour of the lower branch of the neutral curve, which requires the use of inner expansions for the solutions of viscous type, is of the form given by equation (8.25) with the numerical coefficients replaced by 2.296 and 1.002 respectively.

It may appear somewhat surprising that a theory based entirely on the use of outer expansions should yield such good approximations to the lower branch of the neutral curve. This success is apparently due to the fact that $Z \geqslant Z_0$ and that when all expansions are defined in the complete sense Z_0 may be considered large.

One further aspect of the characteristic equation that requires discussion concerns the relation of the foregoing results to the Poincaré form of equation (8.6). If the recessive terms on the left

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hand side of that equation and the balanced and recessive terms on the right-hand side are neglected, then we obtain the usual Poincaré form

$$\frac{\Phi'(y_1)}{\Phi(y_1)} = \frac{\bar{\phi}_3'(y_1)}{\bar{\phi}_3(y_1)}. \tag{8.26}$$

This equation can also be written in the more explicit form

$$\frac{c}{U_1'}\frac{\Phi'(y_1)}{\Phi(y_1)} = -\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}Z^{\frac{3}{2}}+\tfrac{5}{4}+O(Z^{-\frac{3}{2}}\ln Z). \tag{8.27}$$

The corresponding Poincaré form of equation (8.13) is simply $\Delta_1(y_1) = 0$, and on comparing equations (8.13) and (8.27) we see that they differ by a term involving $G_2(y_1)$. This difference is, of course, a direct consequence of the way in which we have ordered the terms in equation (8.13) and would appear to be inevitable. Thus, we regard equation (8.13) as providing a first approximation to the characteristic equation in the complete sense, whereas equation (8.27) provides a second approximation in the Poincaré sense.

9. Results and discussion

To provide a direct comparison between the two approximations to the characteristic equation, equations (8.13) and (8.26), we have made a calculation of the neutral curve for plane Poiseuille flow. For this flow we have $U(y) = 1-y^2$, $y_1 = -1$, $y_c = -(1-c)^{\frac{1}{2}}$ and $y_2 = 0$. We also have

$$G_2(y_1) = -\mathrm{i}U_c'^{\frac{1}{2}}\left\{\frac{101}{24}c^{-\frac{3}{2}}+\frac{23}{24}c^{-\frac{1}{2}}+\frac{23}{24}\frac{c^{\frac{1}{2}}}{1-c}+\tfrac{1}{2}\alpha^2\ln\frac{1+\surd c}{\surd(1-c)}\right\} \tag{9.1}$$

and
$$\hat{z} = (\alpha R)^{\frac{1}{3}}\{\tfrac{3}{4}\mu(c)\}^{\frac{2}{3}} \quad \text{where} \quad \mu(c) = \surd c-(1-c)\ln\frac{1+\surd c}{\surd(1-c)}. \tag{9.2}$$

The computational procedure used to solve equation (8.13) was similar to the one described by Hughes & Reid (1968) and need not therefore be discussed in detail. The results are shown in figure 4 where they are also compared with the neutral curve obtained from equation (8.26) (Reid 1965).

Both approximations to the characteristic equation must, of course, yield an asymptote to the upper branch which is exact, but the asymptotes to the lower branch are vastly different. The lower asymptote obtained from equation (8.26) has long been known to be spurious and it is perhaps even accidental that equation (8.26) leads to a lower branch at all. On the other hand, the lower asymptote obtained from equation (8.13) is virtually indistinguishable from the exact result but such close agreement must also be considered at least partly accidental. On the whole, however, it would appear that a first approximation to the characteristic equation in the complete sense provides a substantially better approximation to the neutral curve than a second approximation in the Poincaré sense.

Finally, we should like to add a few remarks concerning the comparison equation method itself. As previously mentioned in the Introduction, this method has been developed to provide approximations that are uniformly valid in a bounded domain containing one critical point and to provide an algorithm for obtaining higher approximations. While these aims have perhaps been achieved in a formal sense, the method has not heretofore been applied in any great detail and the present analysis suggests some important limitations on it. The uniformity requirement

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can, of course, be satisfied by using the exact solutions of the comparison equation for which integral representations are known. But the approximations which result from this procedure have a very complicated form and are of only theoretical interest. If, however, we relax the uniformity requirement as we have done in this paper, then it becomes of great importance to

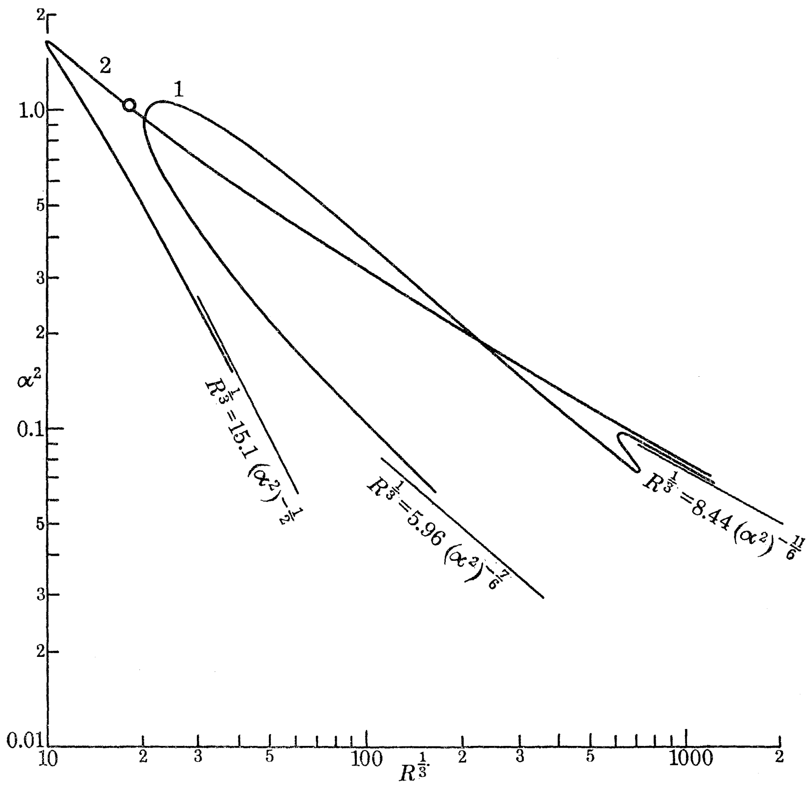

FIGURE 4. A comparison of the curves of neutral stability obtained from a first approximation to the characteristic equation in the complete sense (curve 1) and from a second approximation in the Poincaré sense (curve 2). The circled point denotes the exact value of the minimum critical Reynolds number obtained by direct numerical integration (Reynolds & Potter 1967).

consider Olver's concept of completeness. Ideally this would lead to a theory of error bounds which would then provide a rigorous justification for the present theory. Even after having relaxed the uniformity requirement in this way, the problem of obtaining higher approximations still remains quite formidable and requires the determination not only of higher approximations to the solutions of the comparison equation but also of additional coefficients in the expansion (3.4). These difficulties with the comparison equation method have often prompted the hope that simpler methods of approximation could be devised which would lead to approximations valid in a bounded domain containing one critical point and which would permit the easy and systematic determination of higher approximations. By using the method of matched asymptotic expansions along the lines discussed by Eagles (1969), together with the idea of completeness discussed in this paper, we believe that a relatively simple theory can be developed which will meet these requirements.

We are grateful to Dr T. H. Hughes for providing us with his calculations of $f(Z)$. The research reported in this paper has been supported in part by the National Science Foundation under grant no. GP-8620.

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Appendix. The solutions of the comparison equation

In this appendix we wish to define certain standard solutions of the equation

$$\epsilon^3 u^{\mathrm{iv}} - (\eta u'' + \alpha u' + \beta u) = 0, \tag{A 1}$$

which serves as the comparison equation for the present theory. This equation has been studied previously by Rabenstein (1958) for fixed values of α and β. In the present discussion, however, we shall let α and β depend on ϵ in the manner [cf. equations (3.2)]

$$\alpha = \alpha_1 \epsilon^3 + \dots \quad \text{and} \quad \beta = \beta_0 + \beta_1 \epsilon^3 + \dots; \tag{A 2}$$

for it is this dependence of α on ϵ which gives the problem its distinctive character. We also wish to have all of the asymptotic expansions valid in the complete sense of Olver and this requires an appropriate restriction on the domains of validity of the expansions. To simplify this discussion, we shall consider only the case of neutral stability for which

$$\operatorname{ph} \beta_0 = 0 \quad \text{and} \quad \operatorname{ph} \epsilon = -\tfrac{1}{6}\pi. \tag{A 3}$$

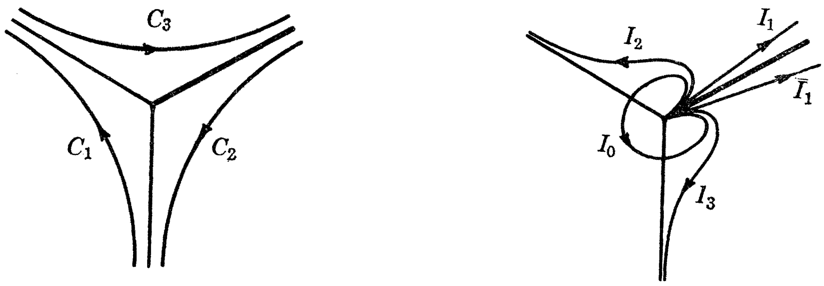

Figure 5. The paths of integration in the t-plane. The heavy lines denote branch cuts.

Exact solutions of equation (A 1) can easily be obtained by the method of Laplace integrals in the form

$$\int_C t^{\alpha-2} \exp\left(\eta t - \tfrac{1}{3}\epsilon^3 t^3 - \beta t^{-1}\right) \mathrm{d}t, \tag{A 4}$$

where the path of integration C must be chosen so that

$$\left[t^{\alpha} \exp\left(\eta t - \tfrac{1}{3}\epsilon^3 t^3 - \beta t^{-1}\right)\right]_C = 0. \tag{A 5}$$

The integrand in the representation (A 4) will, in general, be multiple-valued and it is convenient, therefore, to introduce a cut into the t-plane running from the origin to infinity along the ray $\operatorname{ph} t = \frac{1}{6}\pi$. The admissible paths of integration can then be divided into three types:

(*a*) Three Airy-type paths (C_1, C_2 and C_3) that run from infinity to infinity as shown in figure 5. The solutions associated with these paths are recessive in the sectors $\mathbf{S}_1$, $\mathbf{S}_2$ and $\mathbf{S}_3$ of figure 1 respectively.

(*b*) One path (I_0) that leaves the origin with $\operatorname{Re}(t) > 0$, circles the origin, and then returns to the origin again with $\operatorname{Re}(t) > 0$ as shown in figure 5. The solution associated with this path is well-balanced, i.e. it is balanced for all values of $\operatorname{ph} \eta$.

(*c*) Four paths (I_1, I_2, I_3 and $\bar{I}_1$) that leave the origin with $\operatorname{Re}(t) > 0$ and run to infinity as shown in figure 5. The solutions associated with these paths are purely balanced in the sectors $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$ and $\mathbf{T}_1$ of figure 1 respectively.

We thus have eight solutions associated with these eight paths and, since equation (A 1) is of only the fourth order, they must be related by four exact connexion formulae.

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The solutions $A_k(\eta;\alpha,\beta,\epsilon)$

Consider first the solution associated with the path C_1 and define a standard solution

$$A_1(\eta;\alpha,\beta,\epsilon)$$

by the relation
$$A_1(\eta;\alpha,\beta,\epsilon) = \frac{1}{2\pi i\epsilon}\int_{C_1(t)} t^{\alpha-2}\exp\left(\eta t-\tfrac{1}{3}\epsilon^3t^3-\beta t^{-1}\right)dt. \tag{A 6}$$

To obtain the inner expansion of $A_1(\eta;\alpha,\beta,\epsilon)$ we let $\zeta = \eta/\epsilon$ and set $s = \epsilon t$ so that equation (A 6) becomes
$$A_1(\eta;\alpha,\beta,\epsilon) = \frac{1}{2\pi i}\,\epsilon^{-\alpha}\int_{C_1(s)} s^{\alpha-2}\exp\left(\zeta s-\tfrac{1}{3}s^3-\beta\epsilon s^{-1}\right)ds, \tag{A 7}$$

where $C_1(s)$ is the usual Airy function path in the s-plane. Next expand $e^{-\beta\epsilon/s}$ in a power series in powers of $1/s$. This series converges in any domain bounded away from the origin and, since the remainder of the integrand is uniformly bounded on $C_1(s)$, we may therefore integrate term-by-term to obtain
$$A_1(\eta;\alpha,\beta,\epsilon) = \frac{1}{2\pi i}\,\epsilon^{-\alpha}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(\beta\epsilon)^n\int_{C_1(s)} s^{\alpha-n-2}\exp\left(\zeta s-\tfrac{1}{3}s^3\right)ds. \tag{A 8}$$

On further expanding s^{α} and β^n according to equations (A 2) we obtain the required inner expansion in the form
$$A_1(\eta;\alpha,\beta,\epsilon) = \epsilon^{-\alpha}\{A_1(\zeta,1)+\beta_0\epsilon A_1(\zeta,2)+\tfrac{1}{2}\beta_0^2\epsilon^2A_1(\zeta,3)+O(\epsilon^3)\}, \tag{A 9}$$

where (cf. Hughes & Reid 1968)
$$A_1(\zeta,p) = \frac{1}{2\pi i}\exp[(p+1)\pi i]\int_{C_1(s)} s^{-p-1}\exp\left(\zeta s-\tfrac{1}{3}s^3\right)ds. \tag{A 10}$$

The outer expansions of $A_1(\eta;\alpha,\beta,\epsilon)$ must, of course, exhibit the Stokes phenomenon. In the sector $-\frac{5}{6}\pi < \operatorname{ph}\eta < \frac{1}{2}\pi$, the required outer expansion can be obtained either by applying the method of steepest descents to the integral representation (A 6) as Rabenstein (1958) has done or by re-expansion of equation (A 9). On noting that
$$A_1(\zeta,p) = \tfrac{1}{2}\pi^{-\frac{1}{2}}\zeta^{-\frac{1}{4}(2p+3)}\exp\left(-\tfrac{2}{3}\zeta^{3/2}\right)\{1-(\tfrac{41}{48}+p+\tfrac{1}{4}p^2)\zeta^{-\frac{3}{2}}+O(\zeta^{-3})\} \tag{A 11}$$

in the sector $|\operatorname{ph}\zeta| < \frac{2}{3}\pi$, this latter method immediately gives
$$A_1(\eta;\alpha,\beta,\epsilon) = \tfrac{1}{2}\pi^{-\frac{1}{2}}\epsilon^{-\alpha+\frac{5}{4}}\eta^{-\frac{5}{4}}\exp\left(-\tfrac{2}{3}\epsilon^{-\frac{3}{2}}\eta^{\frac{3}{2}}\right)\{1-(\tfrac{101}{48}\eta^{-\frac{3}{2}}-\beta_0\eta^{-\frac{1}{2}})\epsilon^{\frac{3}{2}}+O(\epsilon^3)\}. \tag{A 12}$$

The outer expansion of $A_1(\eta;\alpha,\beta,\epsilon)$ in the sector $-\frac{3}{2}\pi < \operatorname{ph}\eta < -\frac{5}{6}\pi$ contains dominant, balanced and recessive terms. It cannot be obtained by re-expansion of equation (A 9) since that fails to give the balanced term correctly. It can be obtained, however, either by a second application of the method of steepest descents or, more simply, by use of the connexion formulae which will be given later.

The solutions A_2 and A_3 associated with the paths C_2 and C_3 are defined in a similar manner. They can be expressed in terms of A_1 by means of the relations
$$\left.\begin{aligned} A_2(\eta;\alpha,\beta,\epsilon) &= e^{-\frac{2}{3}\pi i}A_1(\eta\, e^{\frac{2}{3}\pi i};\alpha,\beta\, e^{-\frac{2}{3}\pi i},\epsilon),\\ \text{and}\qquad A_3(\eta;\alpha,\beta,\epsilon) &= e^{\frac{2}{3}\pi i}A_1(\eta\, e^{-\frac{2}{3}\pi i};\alpha,\beta\, e^{\frac{2}{3}\pi i},\epsilon),\end{aligned}\right\} \tag{A 13}$$

and the corresponding inner and outer expansions then follow directly from equations (A 9) and (A 12). For example, the outer expansion of A_2 in the sector $-\frac{3}{2}\pi < \operatorname{ph}\eta < -\frac{1}{6}\pi$ is given by
$$A_2(\eta;\alpha,\beta,\epsilon) = i\tfrac{1}{2}\pi^{-\frac{1}{2}}\epsilon^{-\alpha+\frac{5}{4}}\eta^{-\frac{5}{4}}\exp\left(+\tfrac{2}{3}\epsilon^{-\frac{3}{2}}\eta^{\frac{3}{2}}\right)\{1+(\tfrac{101}{48}\eta^{-\frac{3}{2}}-\beta_0\eta^{-\frac{1}{2}})\epsilon^{\frac{3}{2}}+O(\epsilon^3)\}. \tag{A 14}$$

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The solution $B_0(\eta;\alpha,\beta,\epsilon)$

Consider next the solution associated with the path I_0 and define a standard solution $B_0(\eta;\alpha,\beta,\epsilon)$ by the relation

$$B_0(\eta;\alpha,\beta,\epsilon) = \frac{1}{2\pi i}\int_{I_0(t)} t^{\alpha-2}\exp(\eta t - \tfrac{1}{3}\epsilon^3 t^3 - \beta t^{-1})\,dt. \tag{A 15}$$

For the present purposes we require only the outer expansion of this solution, and by a slight specialization of Rabenstein's results we have

$$B_0(\eta;\alpha,\beta,\epsilon) = (\eta/\beta_0)^{\frac{1}{2}} J_1(2\beta_0^{\frac{1}{2}}\eta^{\frac{1}{2}}) + O(\epsilon^3), \tag{A 16}$$

where $J_1(2\beta_0^{\frac{1}{2}}\eta^{\frac{1}{2}})$ is the usual Bessel function of the first kind.

The solutions $B_k(\eta;\alpha,\beta,\epsilon)$

Consider now the solution associated with the path I_3 and define a standard solution

$$B_3(\eta;\alpha,\beta,\epsilon)$$

by the relation

$$B_3(\eta;\alpha,\beta,\epsilon) = \frac{1}{\pi i}\int_{I_3(t)} t^{\alpha-2}\exp(\eta t - \tfrac{1}{3}\epsilon^3 t^3 - \beta t^{-1})\,dt. \tag{A 17}$$

The solutions B_1, B_2 and $\bar{B}_1$ associated with the paths I_1, I_2 and $\bar{I}_1$ are defined in a similar manner. The inner expansions of these solutions have unusually exotic forms but they are not needed for the present theory and a full discussion of them would be much too lengthy to give here. The outer expansion of B_3 in the sector in which it is purely balanced can be obtained without difficulty, however, by the method described by Rabenstein and we find that

$$B_3(\eta;\alpha,\beta,\epsilon) = -(\eta/\beta_0)^{\frac{1}{2}} H_1^{(2)}(2\beta_0^{\frac{1}{2}}\eta^{\frac{1}{2}}) + O(\epsilon^3) \tag{A 18}$$

in the sector $-\frac{5}{6}\pi < \text{ph}\,\eta < -\frac{1}{6}\pi$, where $H_1^{(2)}(2\beta_0^{\frac{1}{2}}\eta^{\frac{1}{2}})$ is the usual Hankel function. The outer expansion of B_3 in the sectors $\mathbf{T}_1$ and $\mathbf{T}_2$ will be obtained later by use of the connexion formulae.

The exact connexion formulae

By applying Cauchy's theorem to the paths of integration shown in figure 5, we obtain the four exact connexion formulae

$$\left.\begin{aligned} 2\epsilon A_1 - B_2 + B_3 &= -2B_0,\\ 2\epsilon A_2 - B_3 + \bar{B}_1 &= 0,\\ 2\epsilon A_3 - B_1 + B_2 &= 0,\\ \text{and}\qquad \bar{B}_1 &= B_1 e^{+2\pi i\alpha}. \end{aligned}\right\} \tag{A 19}$$

On adding the first three of these equations and then eliminating $\bar{B}_1$, we obtain the additional relation

$$A_1 + A_2 + A_3 = -\epsilon^{-1}\{B_0 - \tfrac{1}{2}(1 - e^{2\pi i\alpha})B_1\}, \tag{A 20}$$

which is especially useful in obtaining the outer expansions of the solutions A_k $(k = 1, 2, 3)$, in the sectors $\mathbf{T}_k$ respectively.

Thus, from equation (A 20) we have

$$\begin{aligned} A_1(\eta;\alpha,\beta,\epsilon) = {}& \tfrac{1}{2}\pi^{-\frac{1}{2}}\epsilon^{-\alpha+\frac{5}{4}}\eta^{-\frac{5}{4}}\exp(-\tfrac{2}{3}\epsilon^{-\frac{3}{2}}\eta^{\frac{3}{2}})\{1 - (\tfrac{101}{48}\eta^{-\frac{3}{2}} - \beta_0\eta^{-\frac{1}{2}})\epsilon^{\frac{3}{2}} + O(\epsilon^3)\}\\ & - \epsilon^{-1}\{(\eta/\beta_0)^{\frac{1}{2}} J_1(2\beta_0^{\frac{1}{2}}\eta^{\frac{1}{2}}) + O(\epsilon^3)\} - i\tfrac{1}{2}\pi^{-\frac{1}{2}}\epsilon^{-\alpha+\frac{5}{4}}\eta^{-\frac{5}{4}}\exp(+\tfrac{2}{3}\epsilon^{-\frac{3}{2}}\eta^{\frac{3}{2}})\\ & \times\{1 + (\tfrac{101}{48}\eta^{-\frac{3}{2}} - \beta_0\eta^{\frac{1}{2}})\epsilon^{\frac{3}{2}} + O(\epsilon^3)\} \end{aligned} \tag{A 21}$$

in the sector $\mathbf{T}_1(-\frac{3}{2}\pi < \text{ph}\,\eta < -\frac{5}{6}\pi)$. The dominant term in this expansion is, of course, simply

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the continuation of equation (A 12). The expansion also contains balanced and recessive terms, however, and is thus complete in the sense of Olver. To this order of approximation, B_1 makes no contribution to the balanced term in the expansion. In a similar way we obtain

$$\begin{aligned} A_2(\eta;\alpha,\beta,\epsilon) = {} & \mathrm{i}\tfrac{1}{2}\pi^{-\frac{1}{2}}\,\epsilon^{-\alpha+\frac{5}{4}}\,\eta^{-\frac{5}{4}}\exp\left(+\tfrac{2}{3}\epsilon^{-\frac{3}{2}}\eta^{\frac{3}{2}}\right)\{1+(\tfrac{101}{48}\eta^{-\frac{3}{2}}-\beta_0\eta^{-\frac{1}{2}})\,\epsilon^{\frac{3}{2}}+O(\epsilon^3)\} \\ & -\epsilon^{-1}\{(\eta/\beta_0)^{\frac{1}{2}}J_1(2\beta_0^{\frac{1}{2}}\eta^{\frac{1}{2}})+O(\epsilon^3)\}-\tfrac{1}{2}\pi^{-\frac{1}{2}}\,\epsilon^{-\alpha+\frac{5}{4}}\,\eta^{-\frac{5}{4}}\exp\left(-\tfrac{2}{3}\epsilon^{-\frac{3}{2}}\eta^{\frac{3}{2}}\right) \\ & \times\{1-(\tfrac{101}{48}\eta^{-\frac{3}{2}}-\beta_0\eta^{-\frac{1}{2}})\,\epsilon^{\frac{3}{2}}+O(\epsilon^3)\} \end{aligned} \tag{A 22}$$

in the sector $T_2\,(-\tfrac{1}{6}\pi < \mathrm{ph}\,\eta < \tfrac{1}{2}\pi)$.

The outer expansions of B_3 in the sectors $\mathbf{T}_1$ and $\mathbf{T}_2$ can be obtained by using the first and second of the connexion formulae (A 19) and we then find without difficulty that

$$B_3(\eta;\alpha,\beta,\epsilon) = -(\eta/\beta_0)^{\frac{1}{2}}H_1^{(2)}(2\beta_0^{\frac{1}{2}}\eta^{\frac{1}{2}})+O(\epsilon^3)+\begin{cases} +2\epsilon A_2(\eta;\alpha,\beta,\epsilon) & (\eta\in\mathbf{T}_1), \\ -2\epsilon A_1(\eta;\alpha,\beta,\epsilon) & (\eta\in\mathbf{T}_2), \\ 0 & (\eta\in\mathbf{T}_3), \end{cases} \tag{A 23}$$

where $-\tfrac{3}{2}\pi < \mathrm{ph}\,\eta < \tfrac{1}{2}\pi$ and, for economy, we have written A_1 and A_2 rather than their outer expansions (A 12) and (A 14). In deriving these results we have also used the fact that

$$B_2-2B_0 = -(\eta/\beta_0)^{\frac{1}{2}}H_1^{(2)}(2\beta_0^{\frac{1}{2}}\eta^{\frac{1}{2}})+O(\epsilon^3)$$

in $\mathbf{T}_2$. Thus, the outer expansion of B_3 is purely balanced in $\mathbf{T}_3$, dominant with balanced terms in $\mathbf{S}_3$ and balanced with recessive terms in $\mathbf{I}-(\mathbf{S}_3\cup\mathbf{T}_3)$.

The solutions $u_k(\eta)$

From the set of eight solutions discussed above we now wish to pick a fundamental set of four, which will be denoted by $u_k(\eta)$, for use in the construction (3.4). Our choice is primarily influenced by the location of the points at which we must impose the boundary conditions. Since, in the case of neutral stability, one boundary point lies in $\mathbf{S}_1$ and the other in $\mathbf{S}_2$ we shall require that the solutions $u_k(\eta)$ be 'numerically satisfactory' in $\mathbf{S}_1\cup\mathbf{S}_2$, i.e. that they differ by the largest possible factor in $\mathbf{S}_1\cup\mathbf{S}_2$ (cf. Miller 1950). A set of solutions which satisfies this requirement is easily seen to be $\{B_0, B_3, A_1, A_2\}$ where we can, of course, add to B_3 an arbitrary multiple of B_0. Thus we define the fundamental set of solutions

$$\left.\begin{aligned} u_1(\eta) &= B_0(\eta;\alpha,\beta,\epsilon), \\ u_2(\eta) &= \beta(2\gamma-1+\log\beta+\pi\mathrm{i})B_0(\eta;\alpha,\beta,\epsilon)+\beta\pi\mathrm{i}B_3(\eta;\alpha,\beta,\epsilon), \\ u_3(\eta) &= A_1(\eta;\alpha,\beta,\epsilon), \\ \text{and}\qquad u_4(\eta) &= A_2(\eta;\alpha,\beta,\epsilon), \end{aligned}\right\} \tag{A 24}$$

where γ is Euler's constant. The outer expansions of u_1 and u_2 can conveniently be written in the form

$$u_1(\eta) = u_1^{(0)}(\eta)+O(\epsilon^3) \tag{A 25}$$

and

$$u_2(\eta) = u_2^{(0)}(\eta)+O(\epsilon^3)+\begin{cases} +2\pi\mathrm{i}\beta_0\,\epsilon A_2(\eta;\alpha,\beta,\epsilon) & (\eta\in\mathbf{T}_1), \\ -2\pi\mathrm{i}\beta_0\,\epsilon A_1(\eta;\alpha,\beta,\epsilon) & (\eta\in\mathbf{T}_2), \\ 0 & (\eta\in\mathbf{T}_3), \end{cases} \tag{A 26}$$

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where
$$u_1^{(0)}(\eta) = \eta \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+1)!} (\beta_0 \eta)^k, \tag{A27}$$

$$u_2^{(0)}(\eta) = 1 + \beta_0 \eta \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\,(k+1)!} \left\{ 2[\psi(k+1)+\gamma] - \frac{k}{k+1} \right\} (\beta_0 \eta)^k - \beta_0 u_1^{(0)}(\eta) \ln \eta \tag{A 28}$$

and $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function. In equation (A 26) we have again written A_1 and A_2 rather than their outer expansions (A 12) and (A 14). With u_1 and u_2 defined in this way, there is then a certain parallelism between $u_1^{(0)}$ and $\chi_1^{(0)}$ and between $u_2^{(0)}$ and $\chi_2^{(0)}$. Thus $u_1^{(0)}$ and $u_2^{(0)}$ satisfy the reduced form of the comparison equation $\eta u'' + \beta_0 u = 0$, their Wronskian has the value $W(u_1^{(0)}, u_2^{(0)}) = -1$ and the regular part of $u_2^{(0)}$ contains no multiple of $u_1^{(0)}$.

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